PhD Public Defense

On the Complexity of Optimization: Curved Spaces and Benign Landscapes

Candidate: Chris Criscitiello

Advisor: Nicolas Boumal

EPFL

OPTIM, Chair of Continuous Optimization

September 12, 2025

Outline

What is optimization?

Optimization algorithm?

Steepest descent

Best possible algorithm?

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Optimization algorithm?

Steepest descent

Best possible algorithm?

Based on:

- "Negative curvature obstructs acceleration for strongly geodesically convex optimization” - C & **Boumal** – COLT’22
- “Curvature and Complexity: Better lower bounds for geodesically convex optimization” - C & **Boumal** – COLT’23
- “Synchronization on circles and spheres with nonlinear interactions” - C, **Rebjock, McRae, Boumal** - under review
- “The sensor network localization problem has benign landscape under mild rank relaxation” - C, **Rebjock, McRae, Boumal** – under review

What is optimization?

Find minimum of function f over $M = \{\text{set of possibilities}\}$

$$\min_{x \in M} f(x)$$

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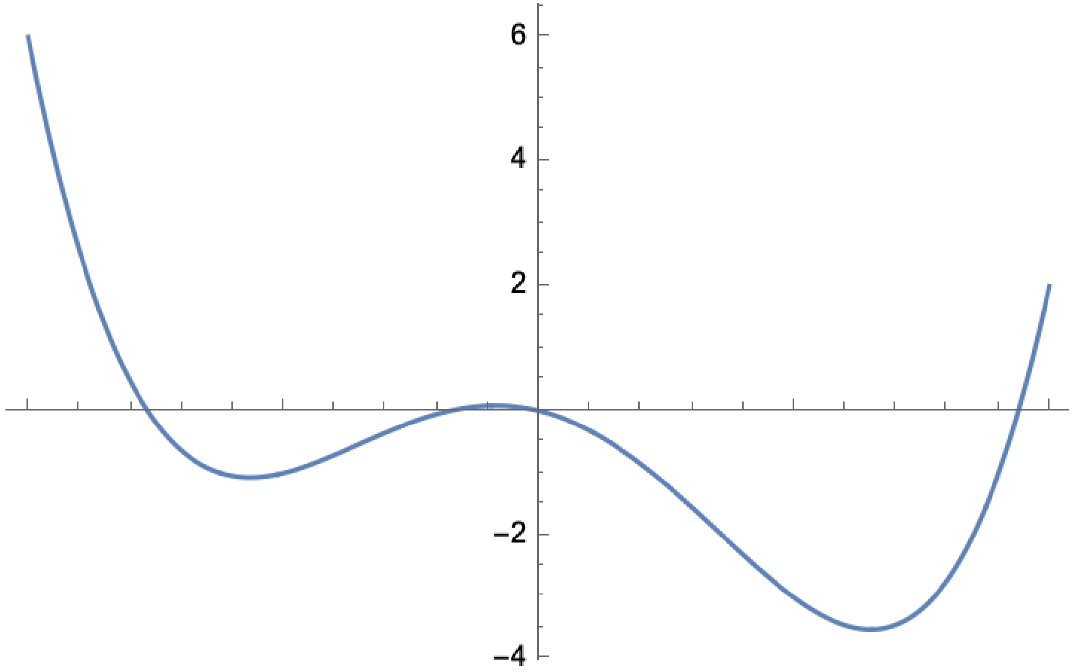
Examples:

$f(x) = x^4 - 3x^2 - x$
$M = \{\text{numbers}\}$



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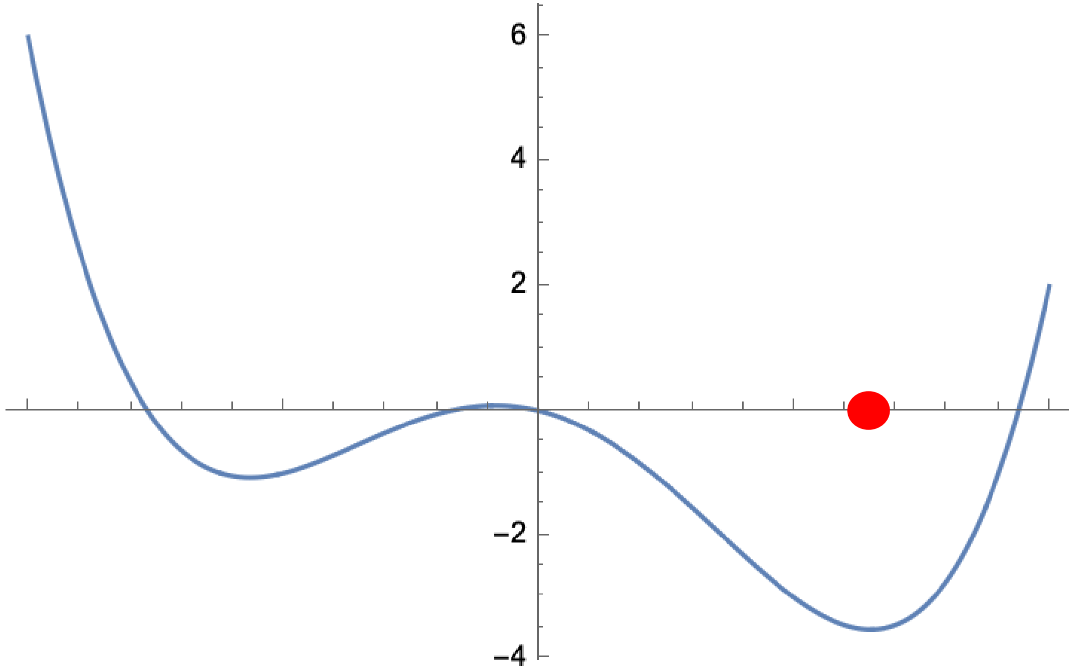

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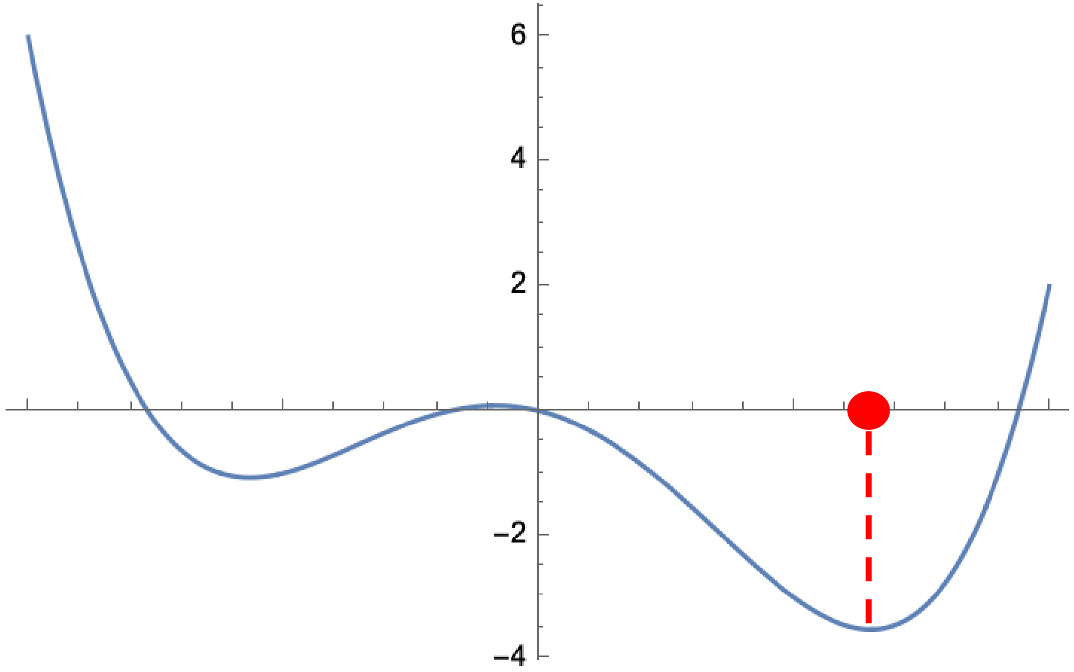

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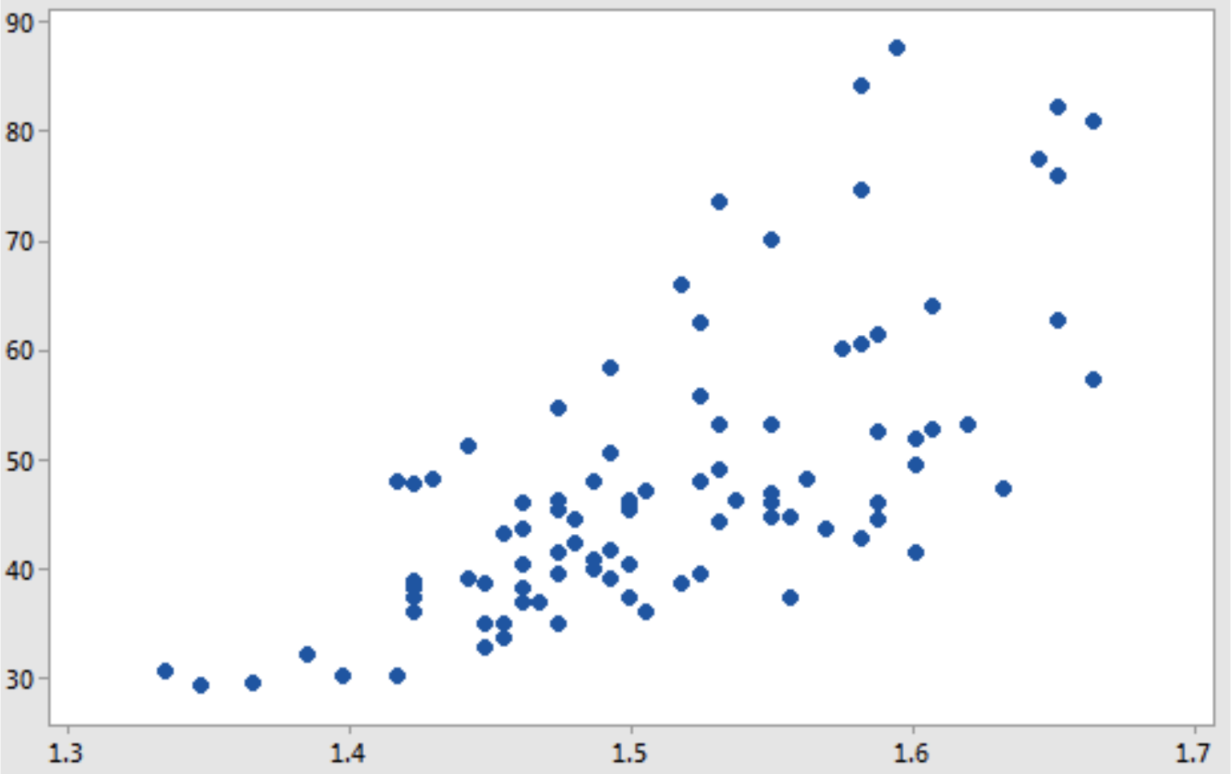

What is optimization?

Find minimum of function f over $M = \{\text{set of possibilities}\}$

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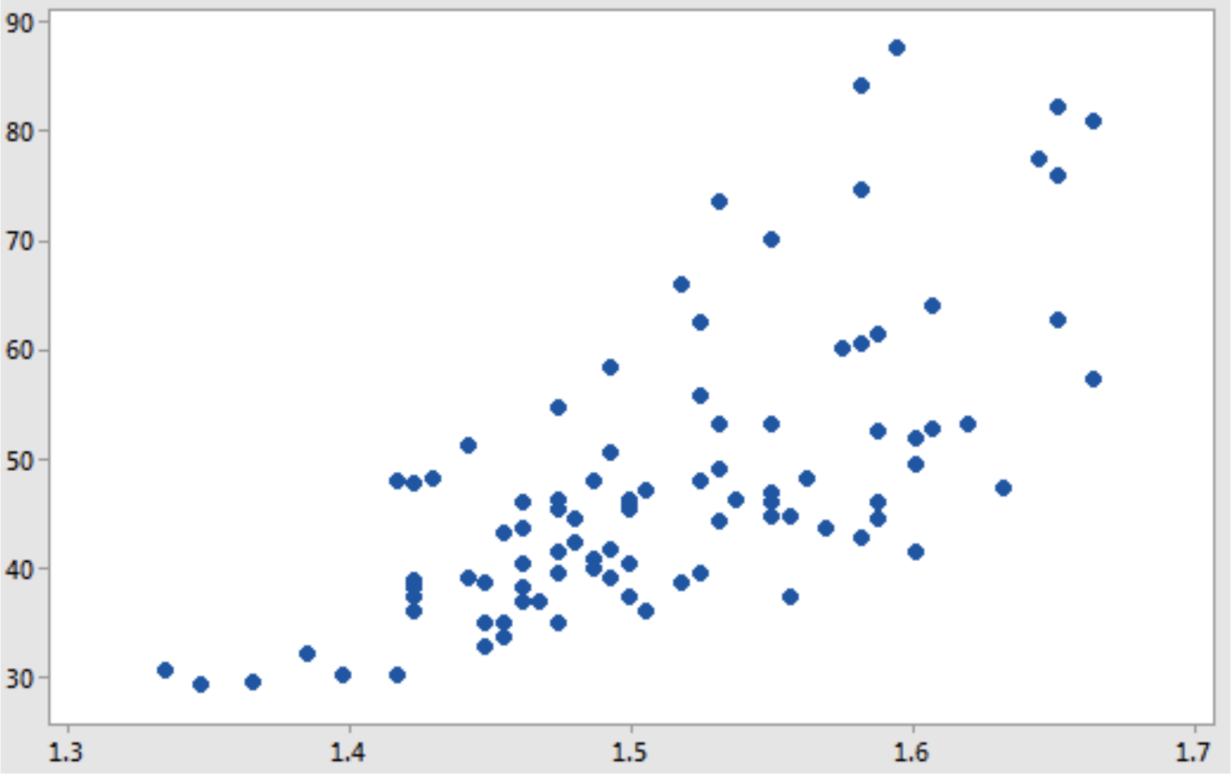
Examples:

$M = \{\text{all lines}\}$

$f(\text{line}) =$ distance between points and line

$$f(\beta) = \|X\beta - y\|^2$$



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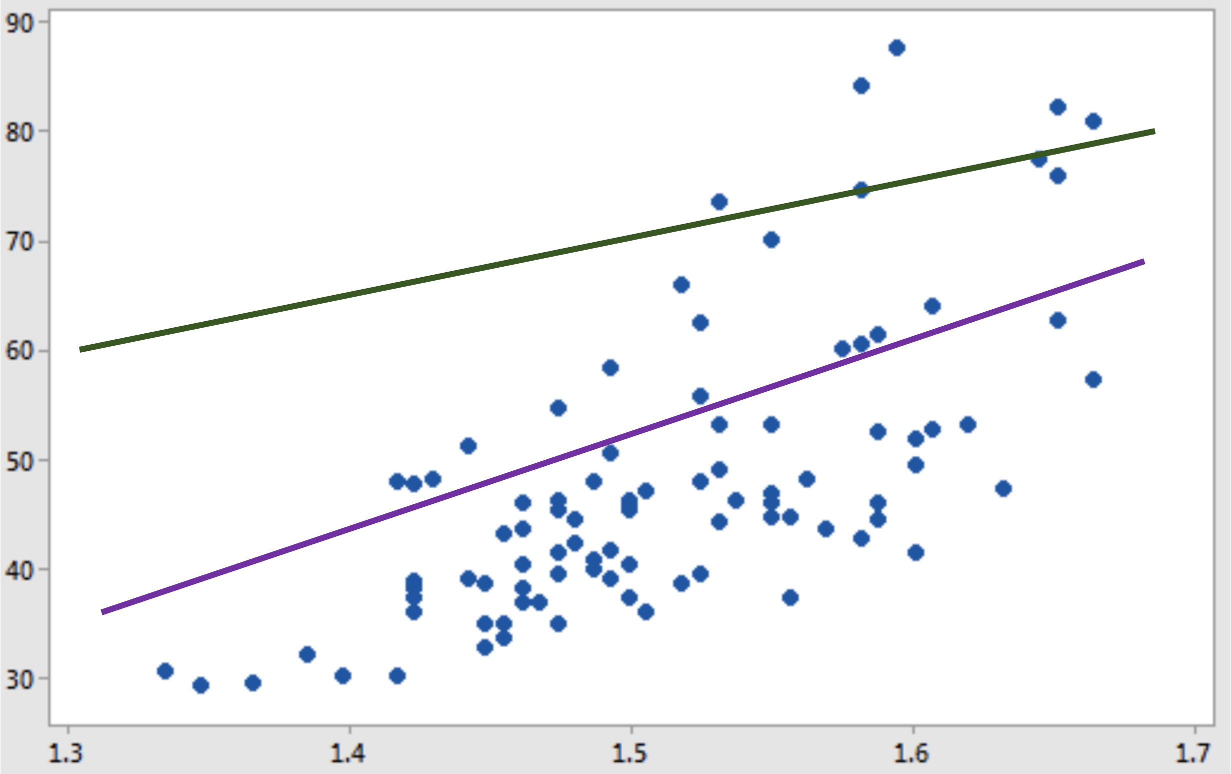

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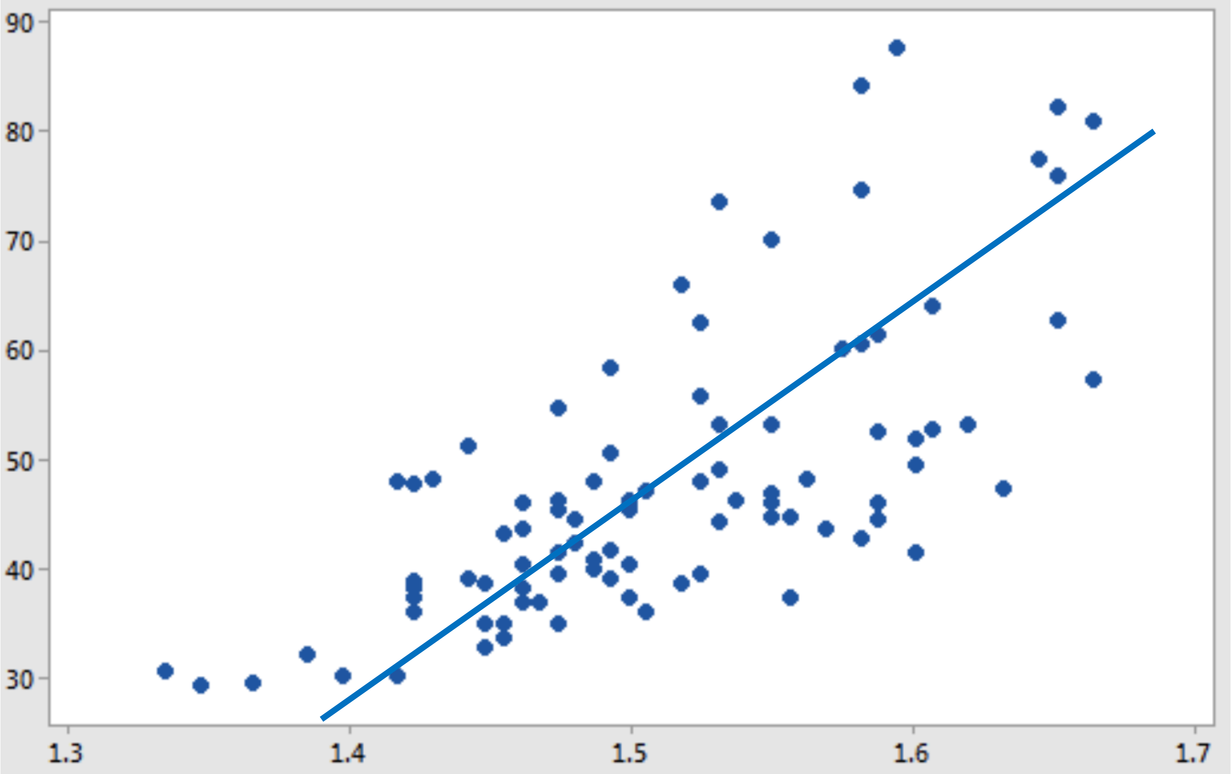

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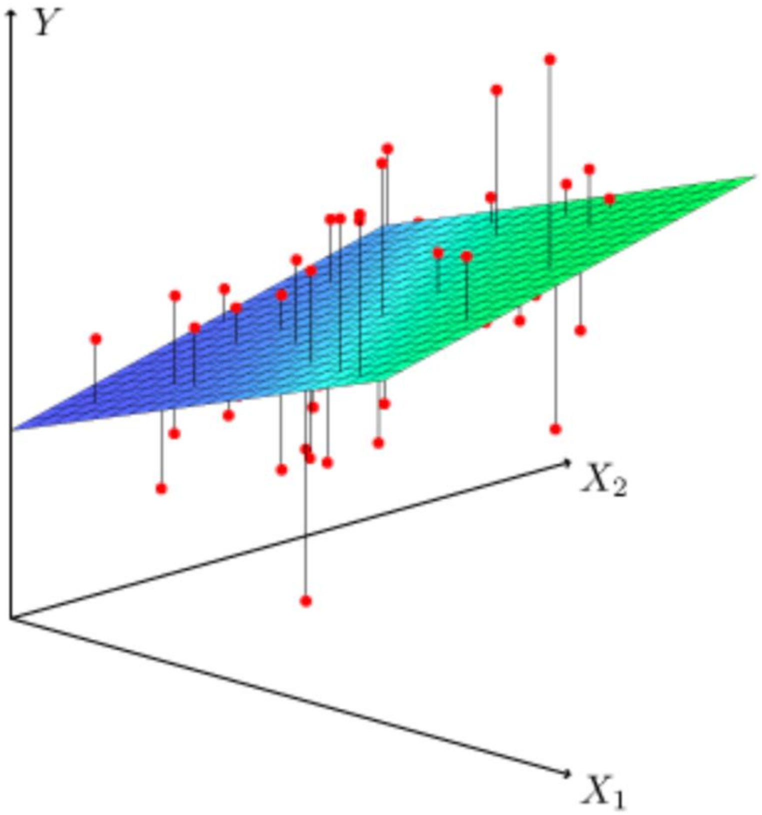
Examples:

Higher dimensional variants

Harder, can't visualize!

Easily encounter **millions** of features



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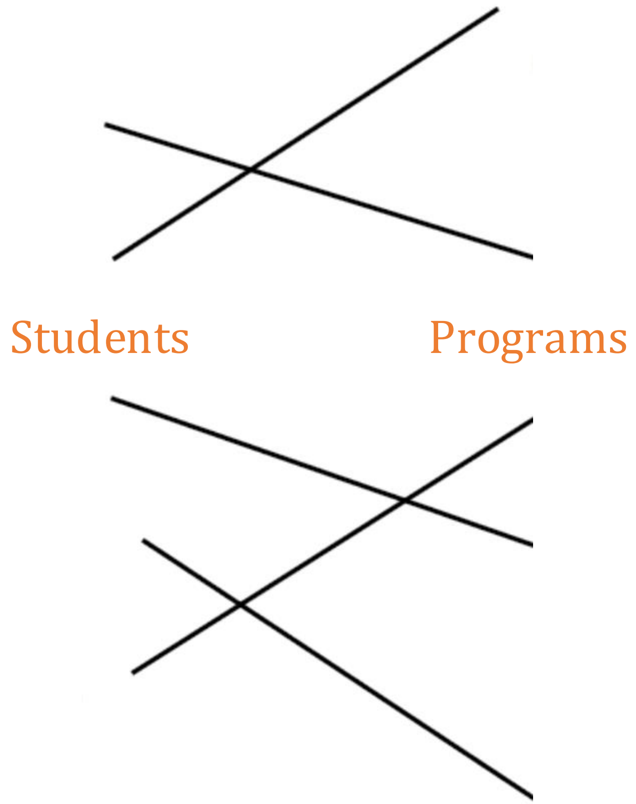
Examples:

How to find *best* matching of medical students to residency programs?

$M = \{\text{all possible matchings}\}$



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- You can evaluate f on an input x, and get $f(x)$

$x \longrightarrow$  $\longrightarrow f(x)$

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- You can evaluate f on an input x, and get $f(x)$
- You can evaluate f on an input x, and get "gradient" $\nabla f(x)$

x  $f(x), \nabla f(x)$

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$\boxed{\text{query}}$ $\quad x \longrightarrow$

$\longrightarrow \quad f(x), \nabla f(x)$

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Algo $\longrightarrow x_2 \longrightarrow$  $\longrightarrow f(x_2), \nabla f(x_2)$

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$\cdots$

x_T so that $f(x_t)$ is small

Steepest descent

gradient $\nabla f(x)$ = direction of steepest descent

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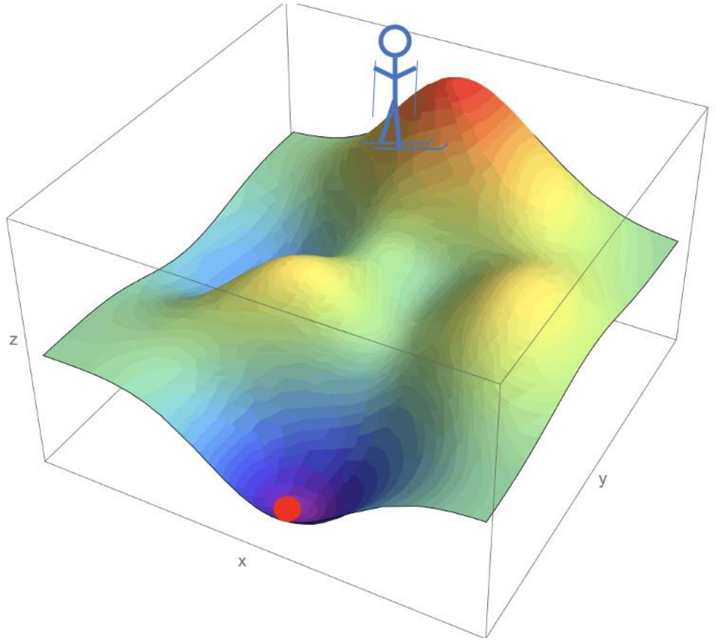

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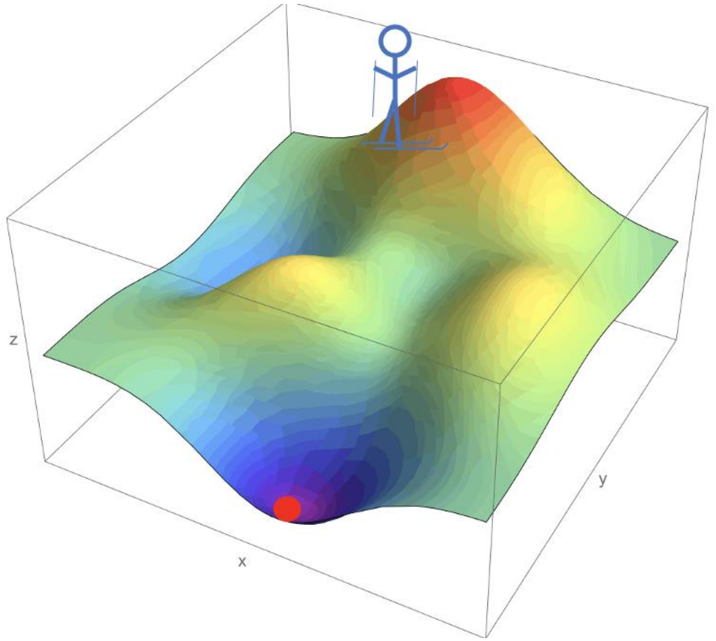
Simple idea: follow direction of steepest descent!



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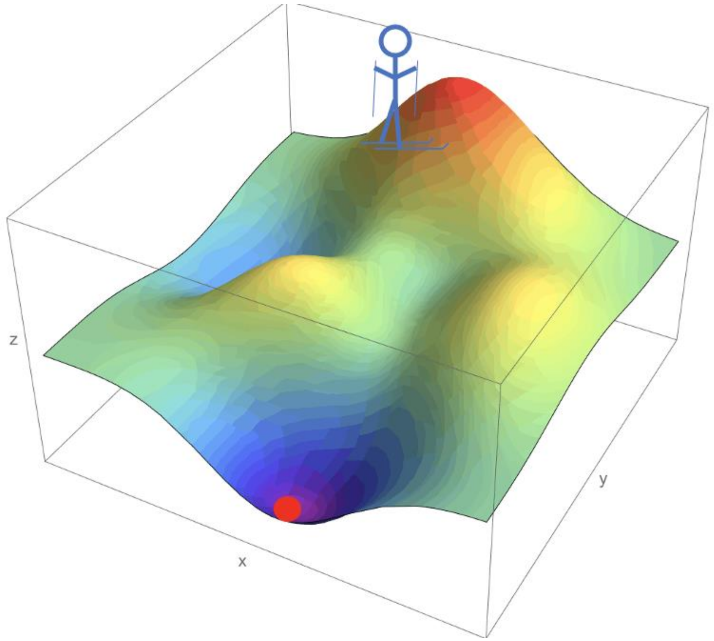
Algorithm = **Steepest Descent**

$$x_{k+1} = x_k - \eta \nabla f(x_k)$$



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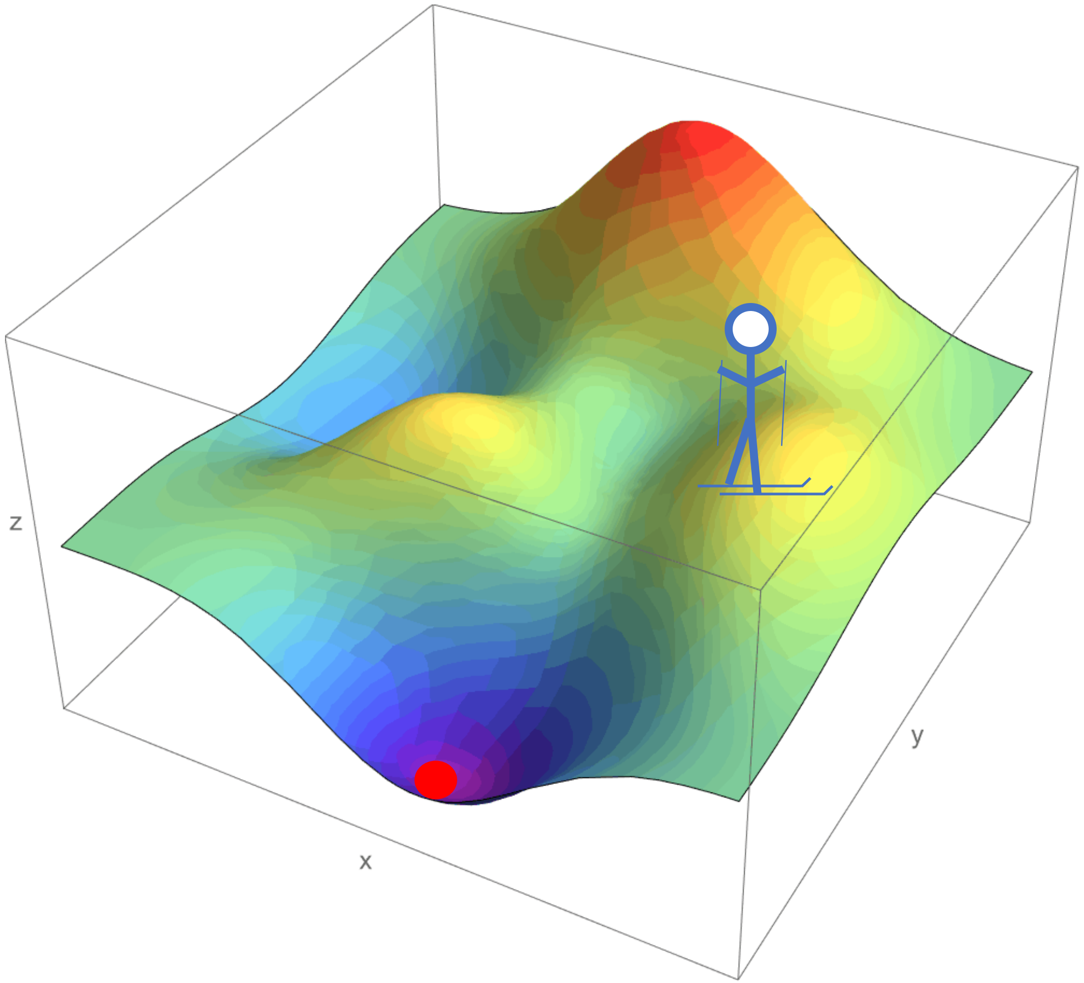

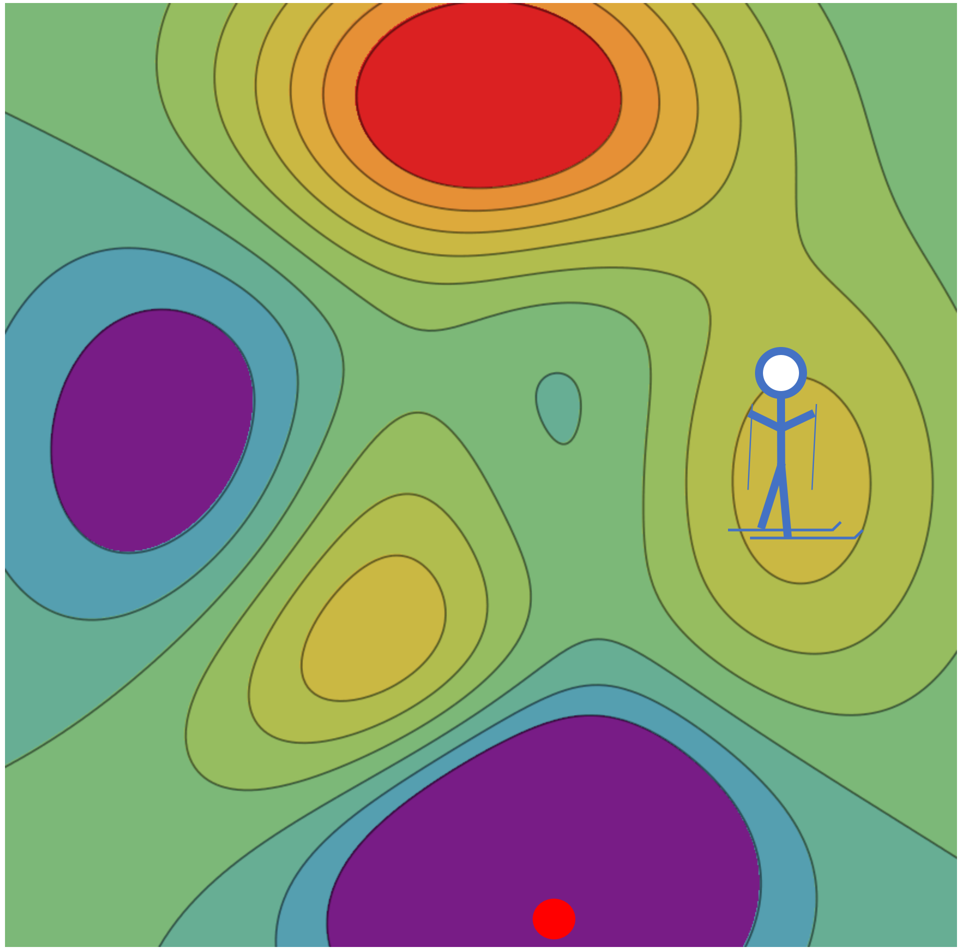

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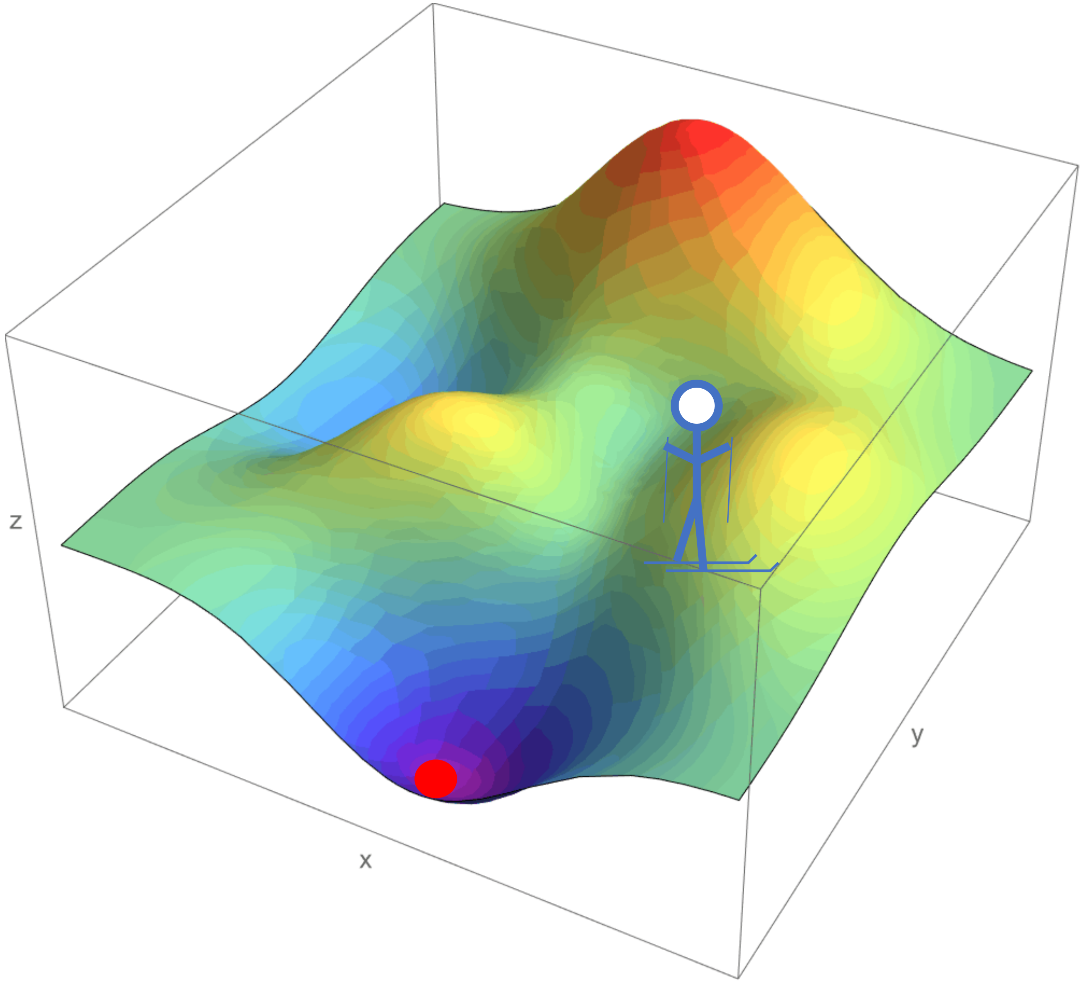

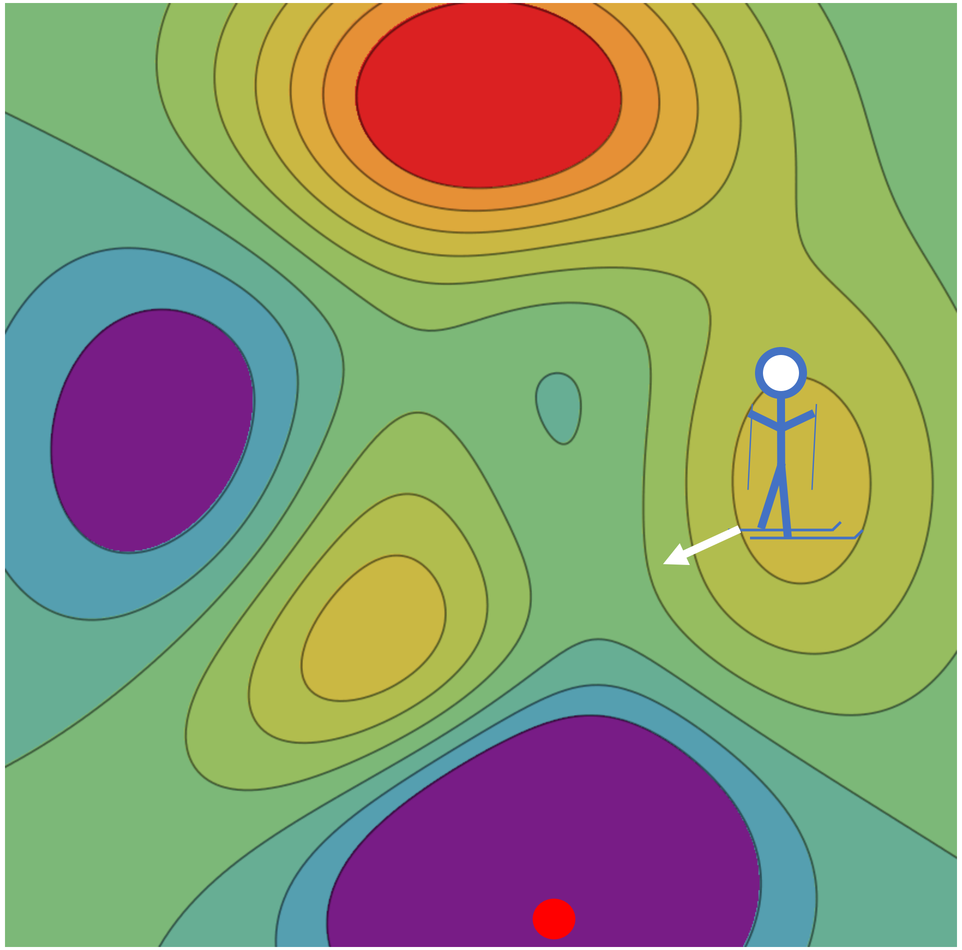

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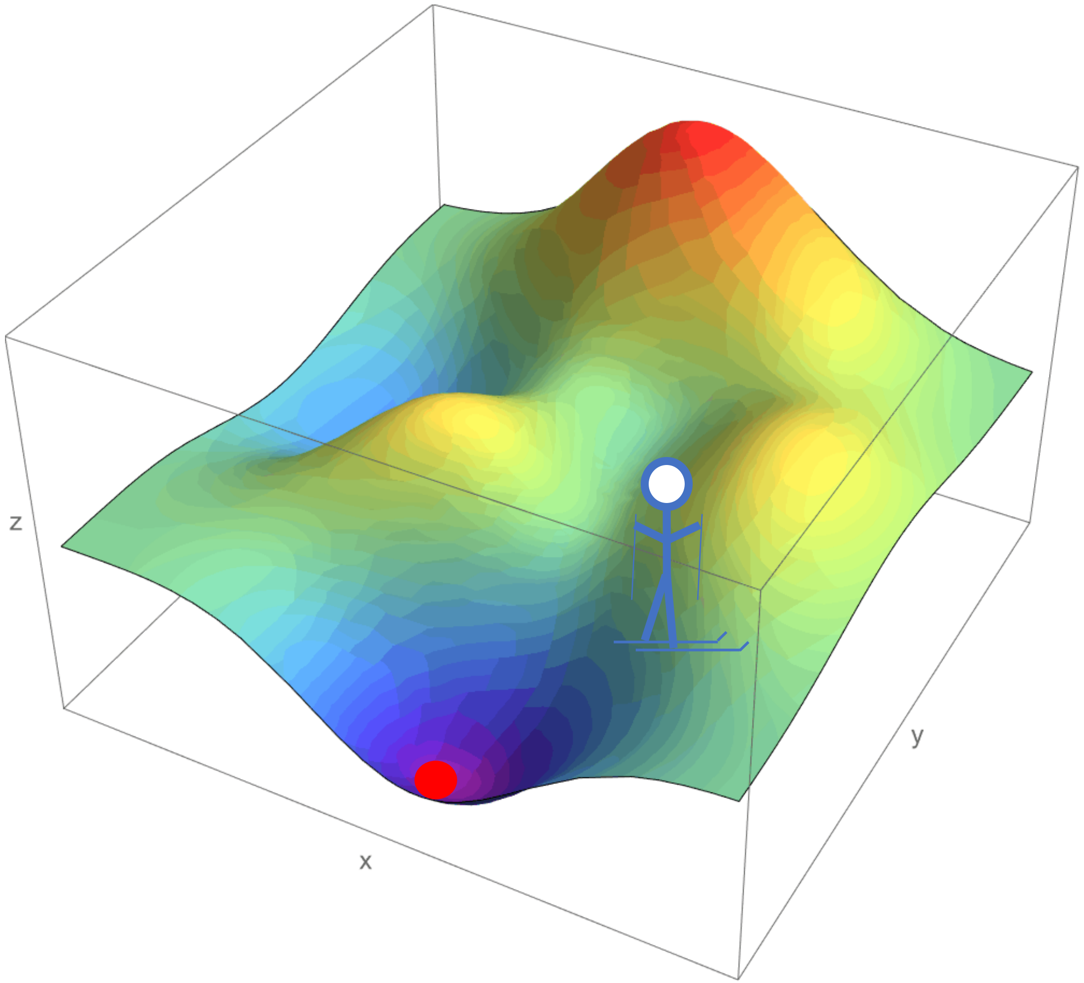

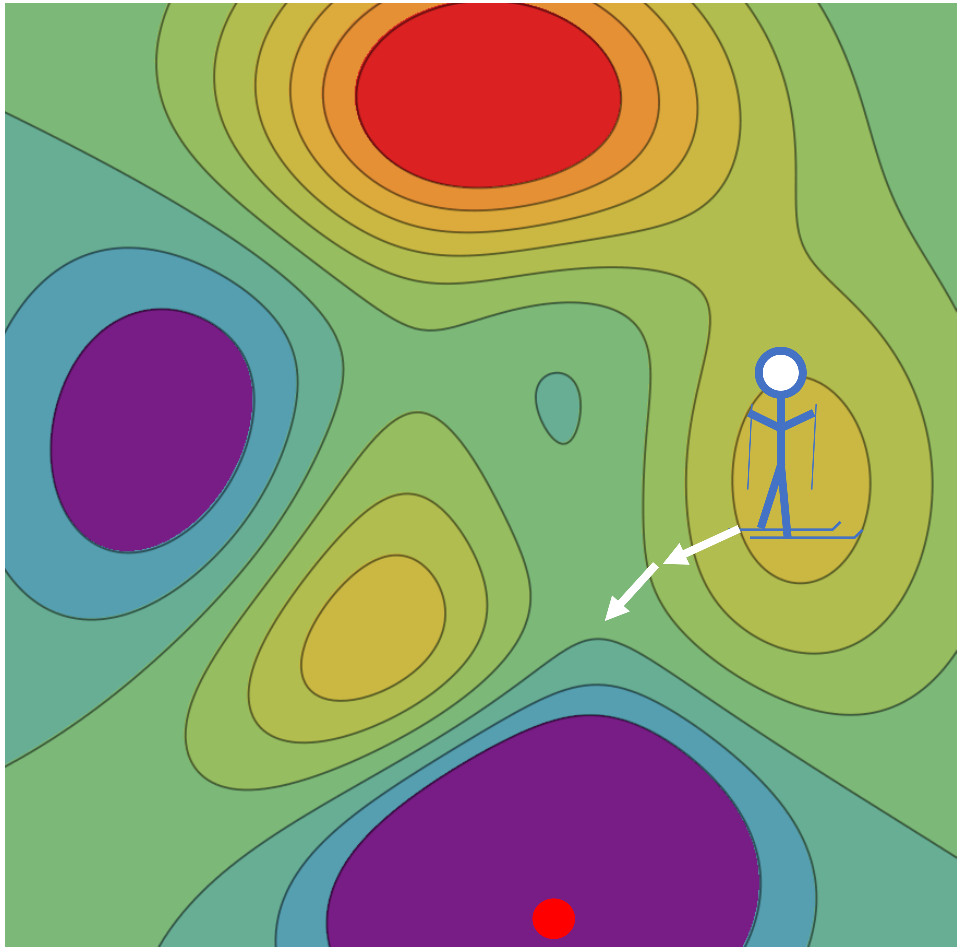

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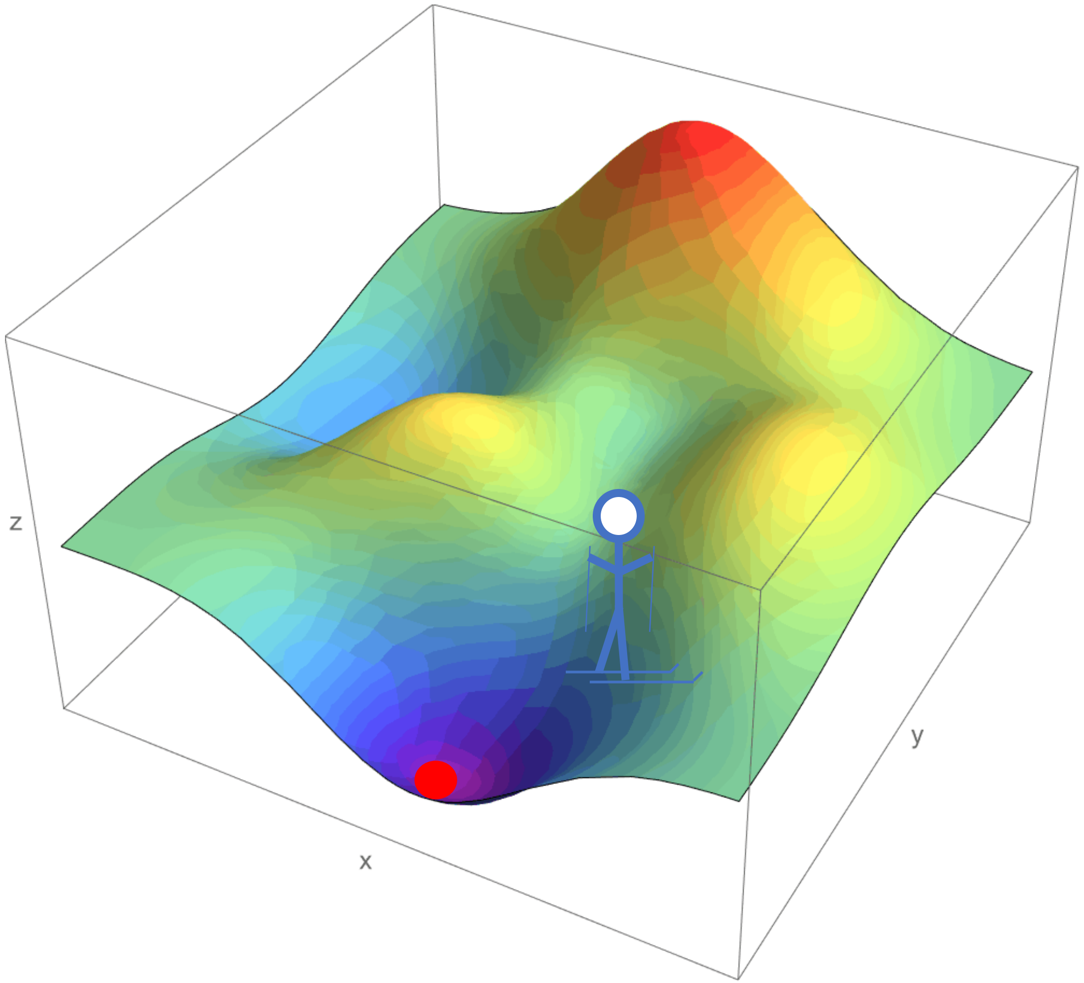

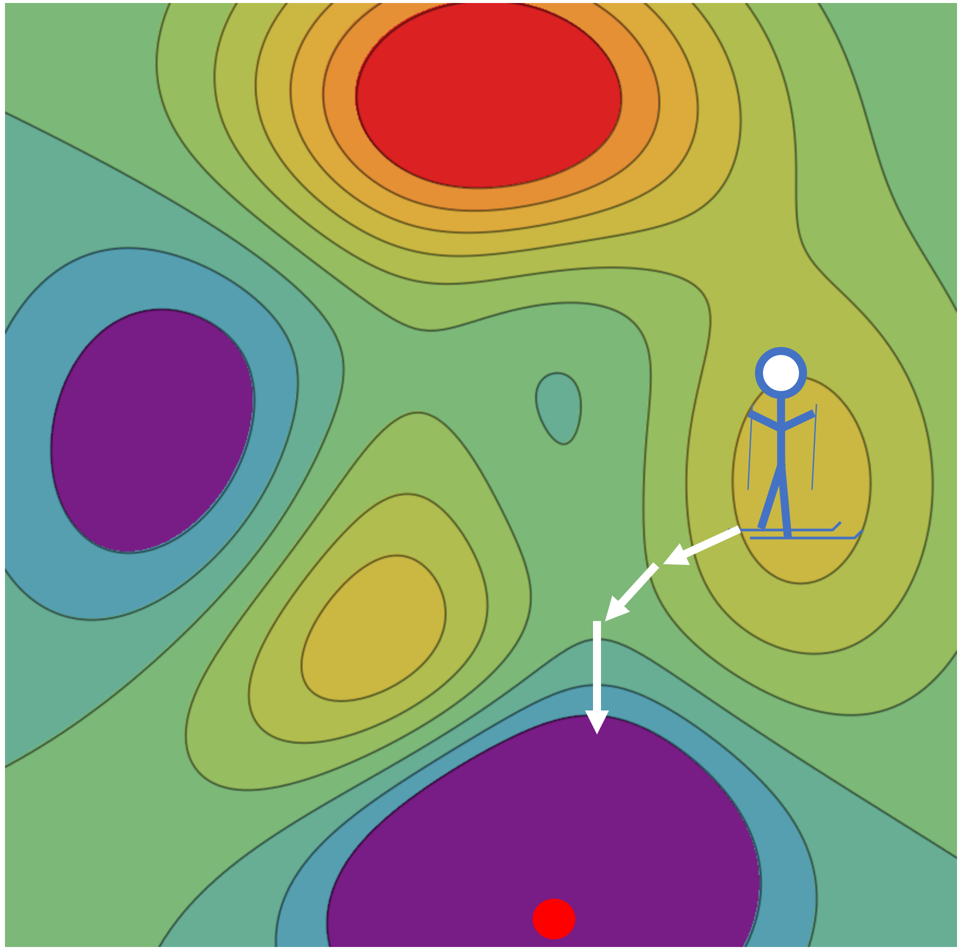

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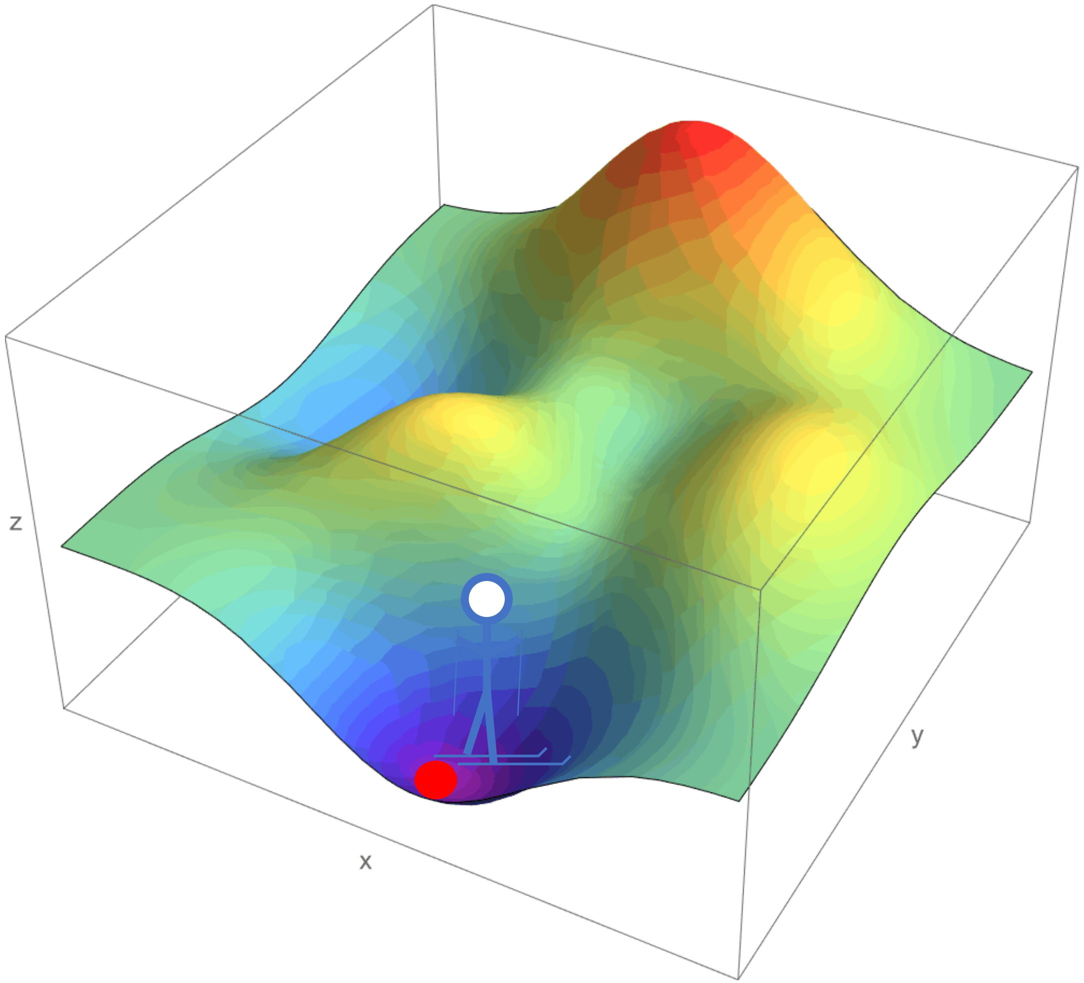

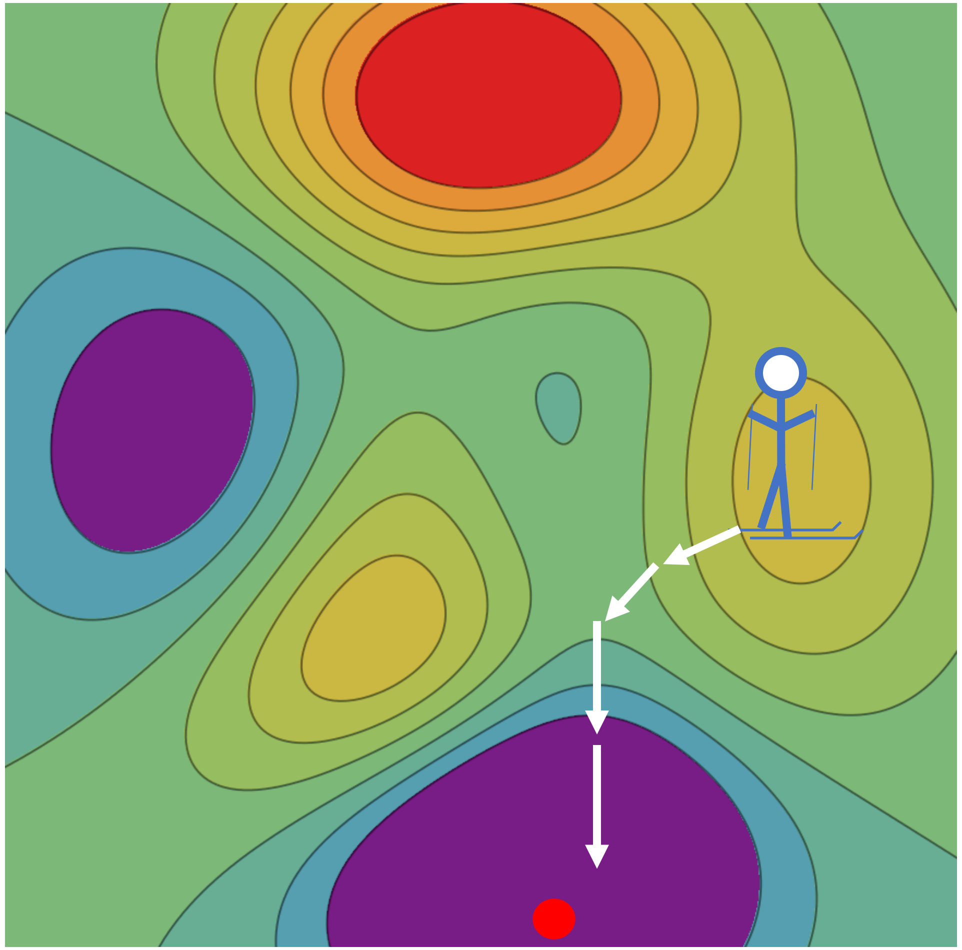

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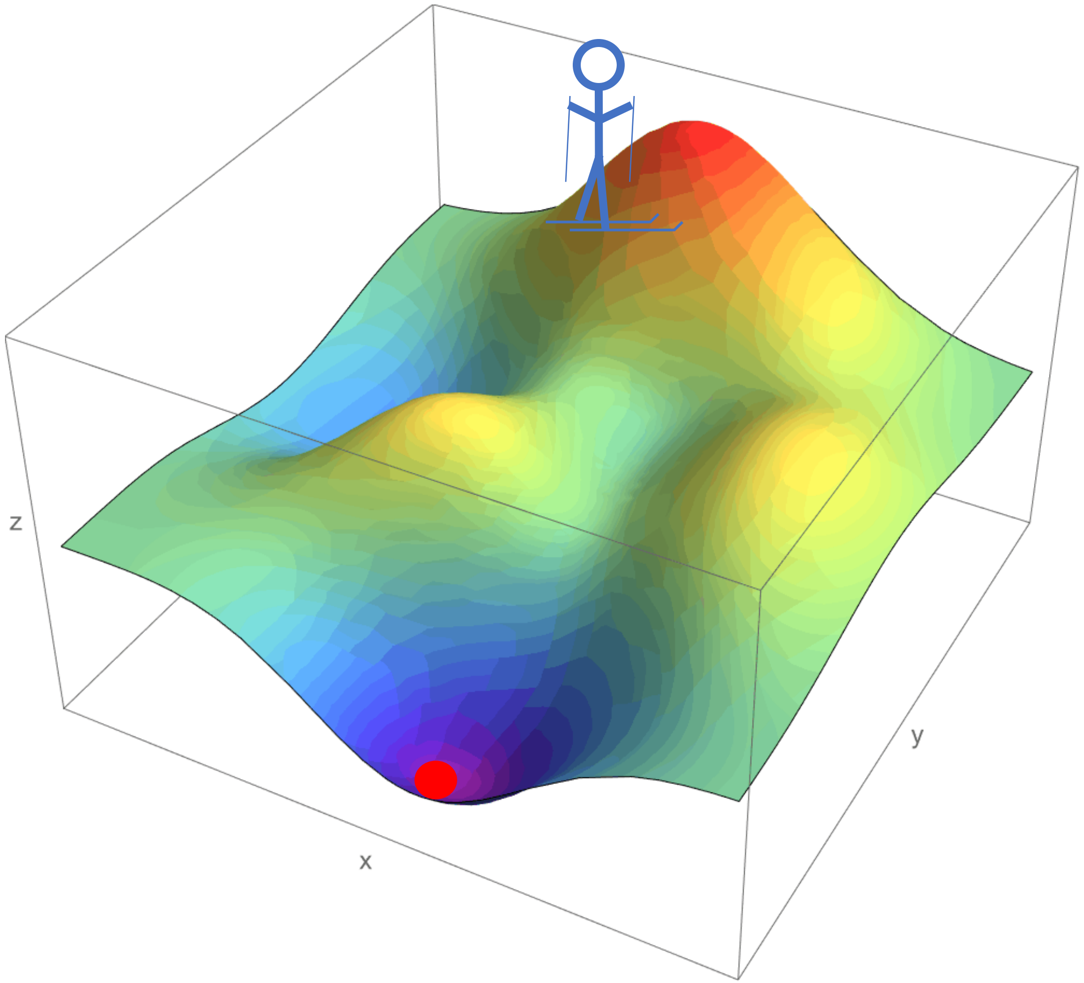

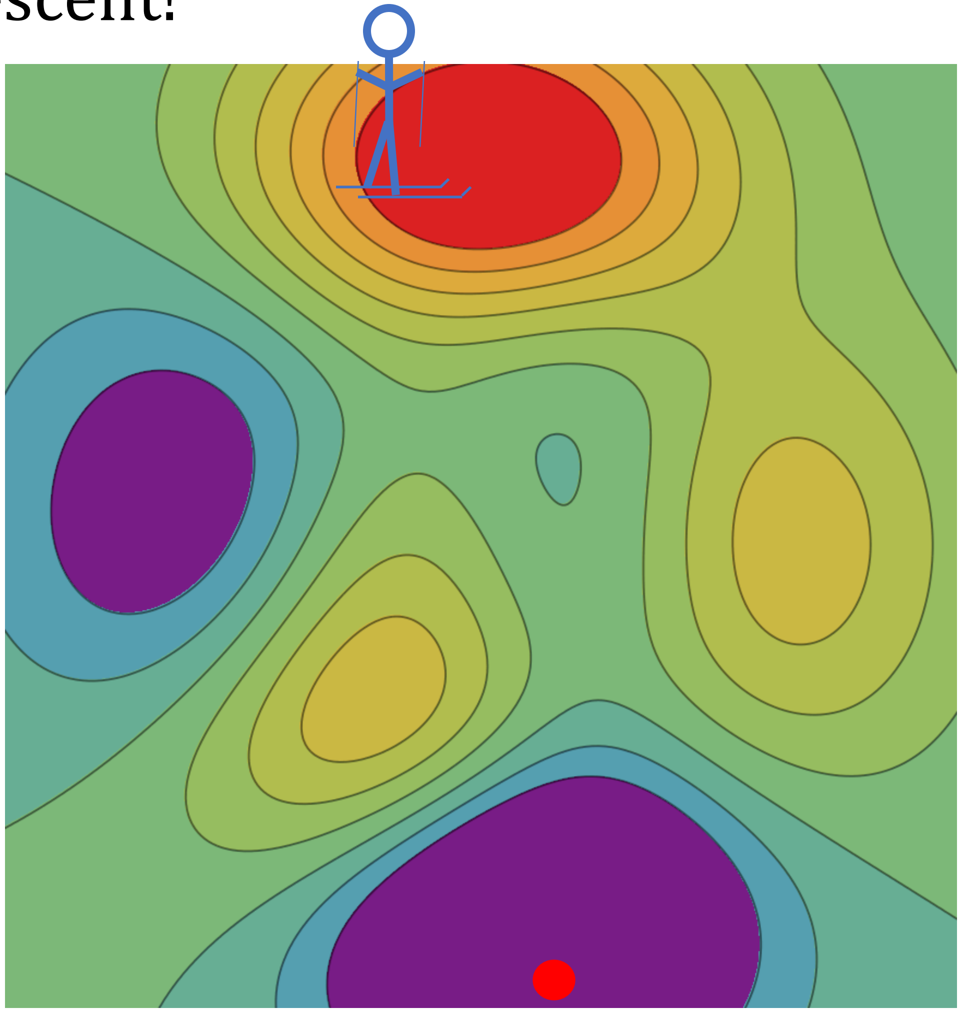

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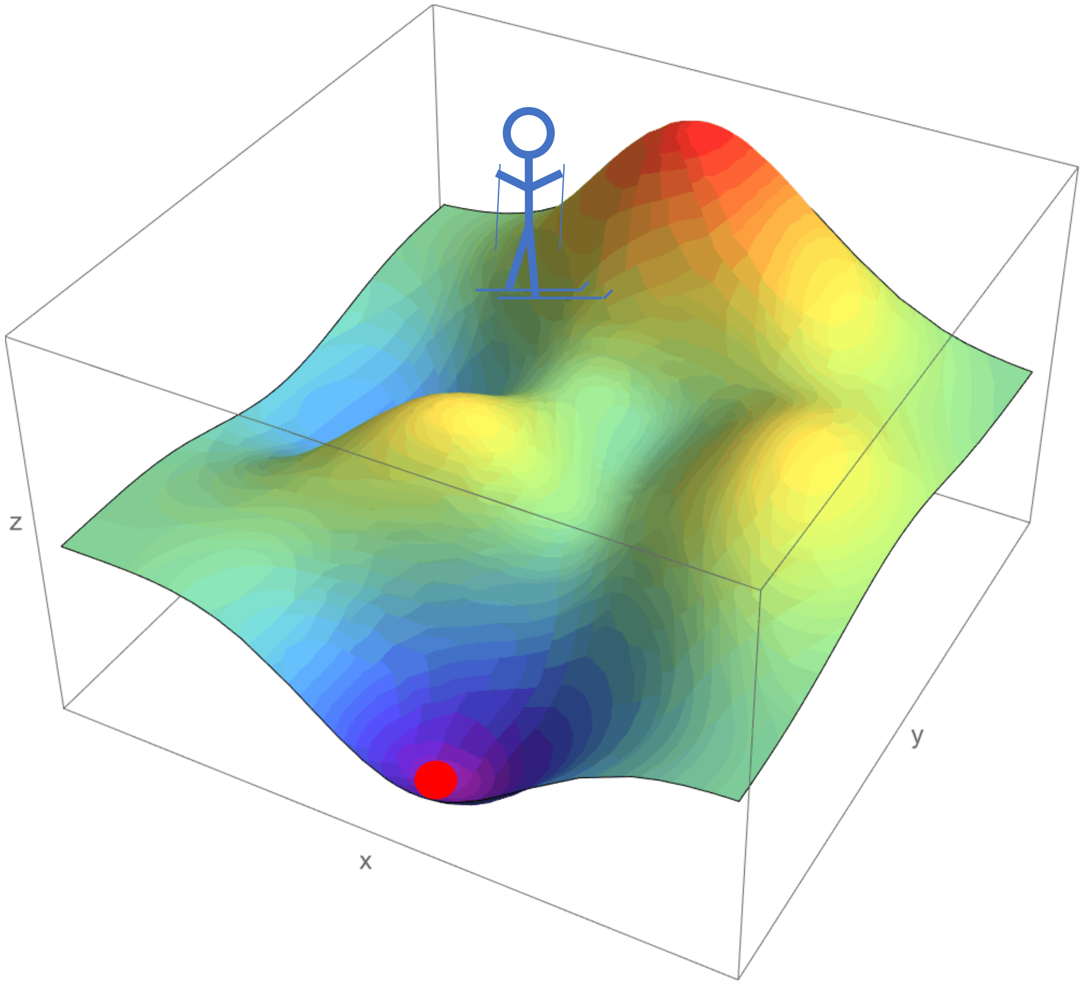

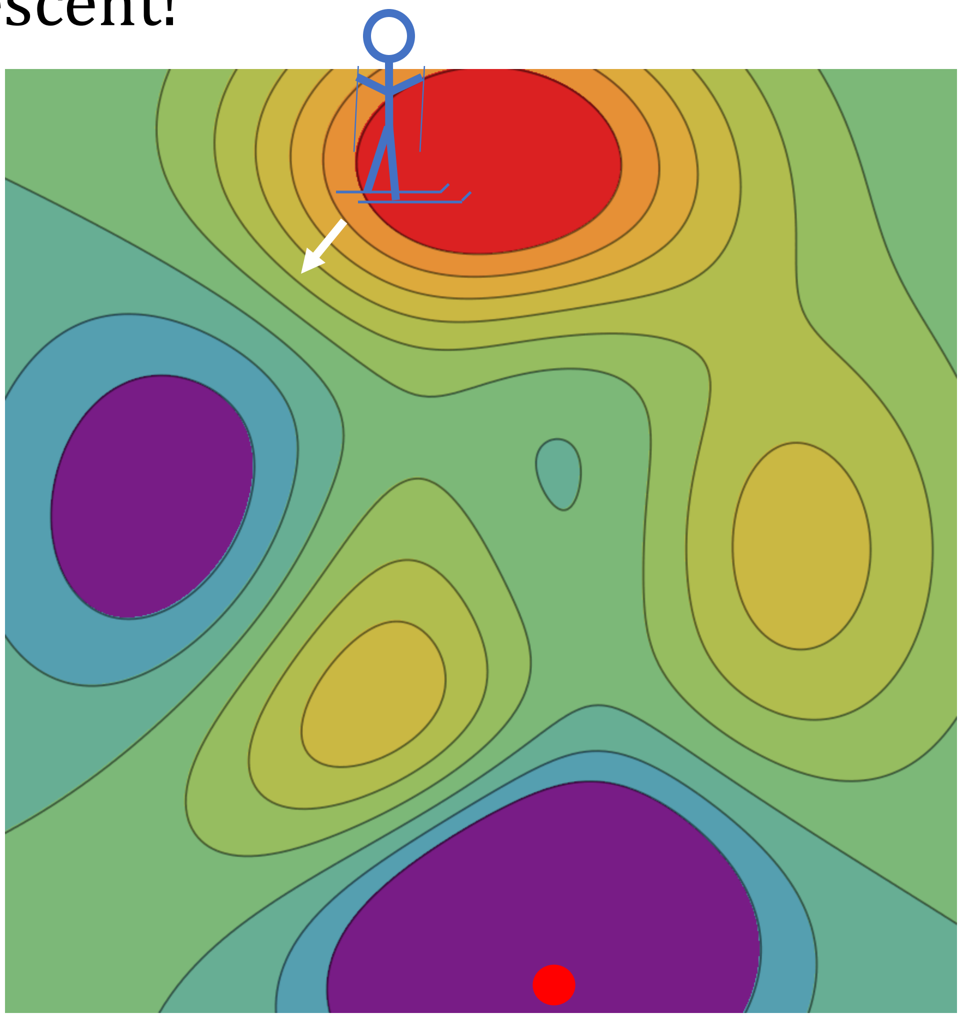

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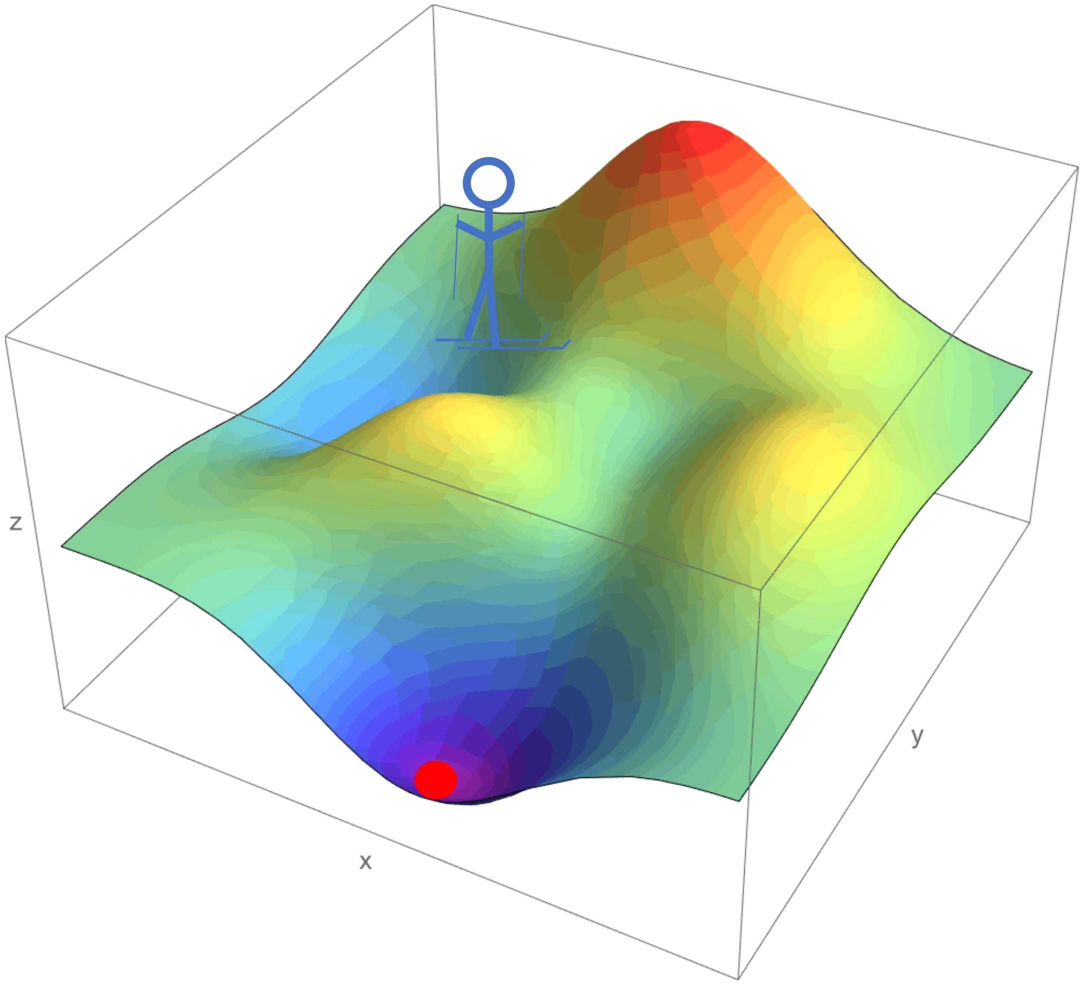

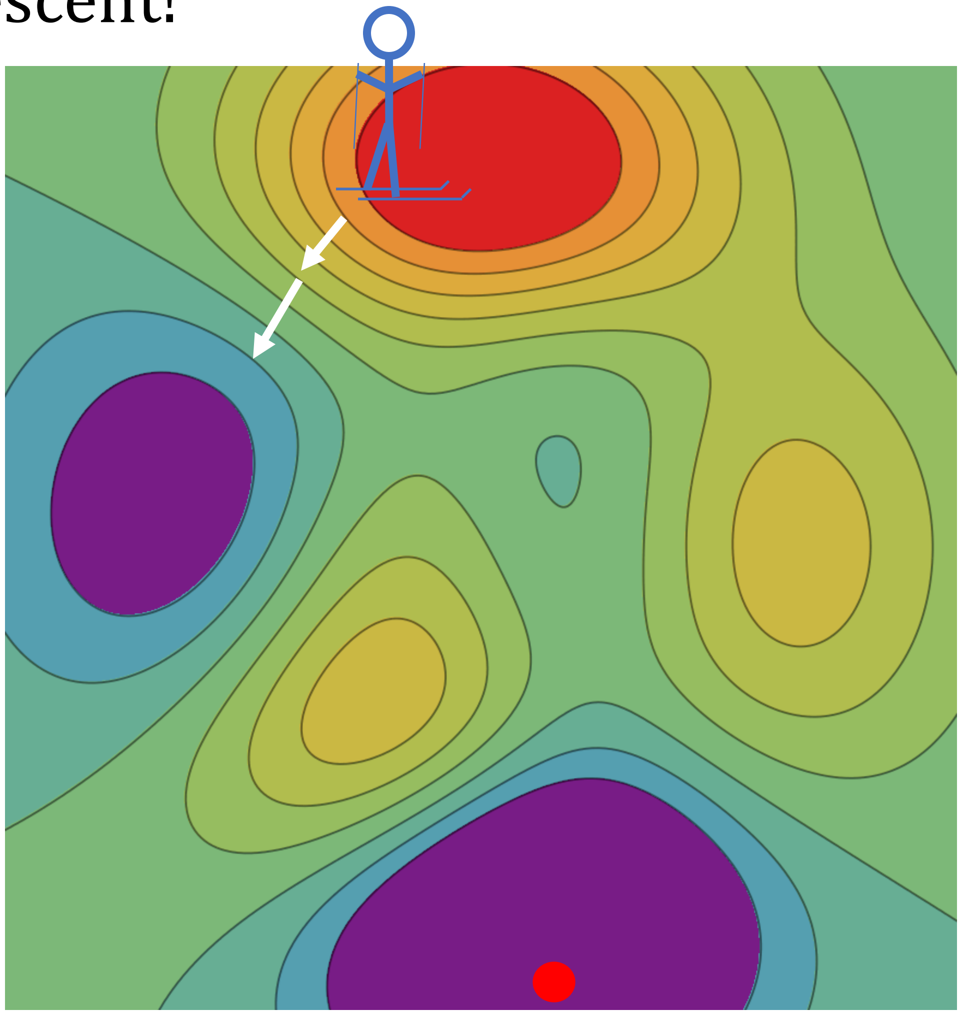

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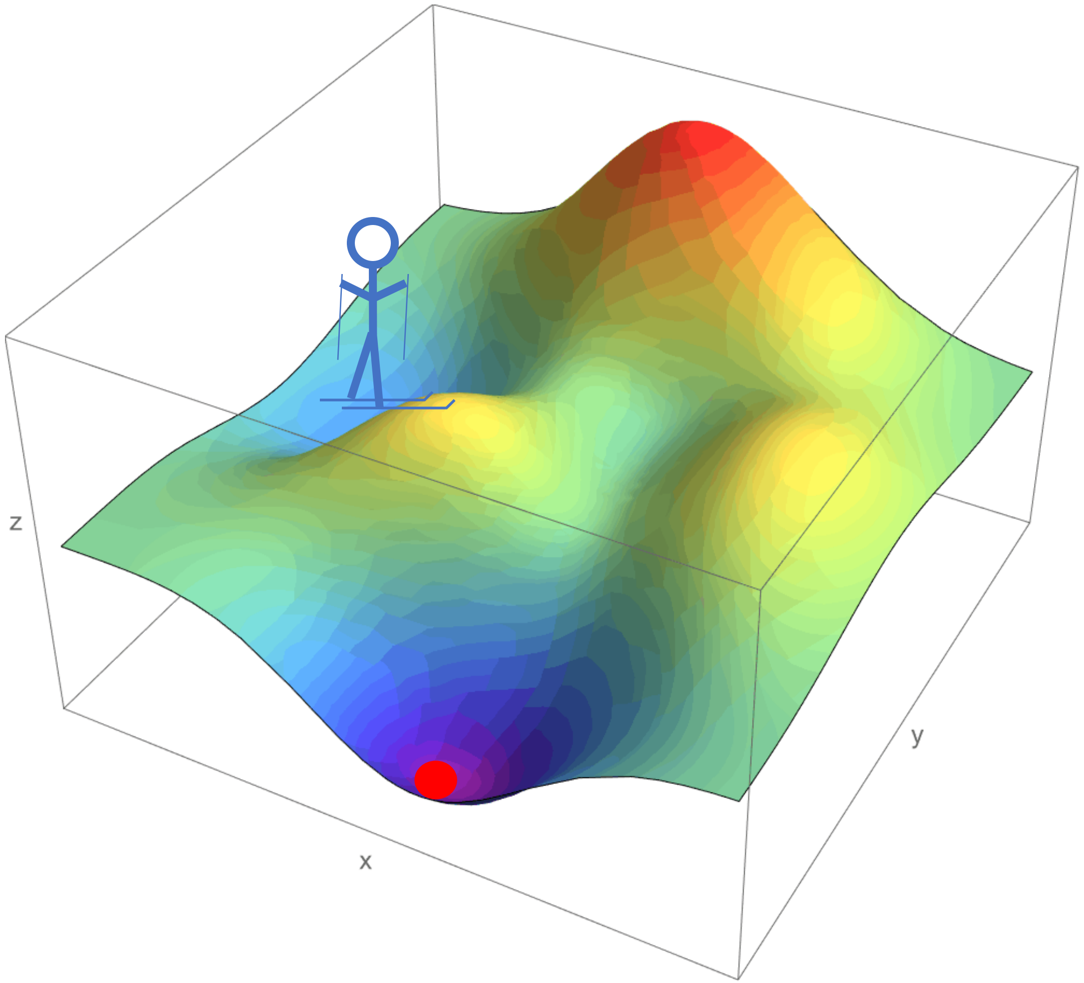

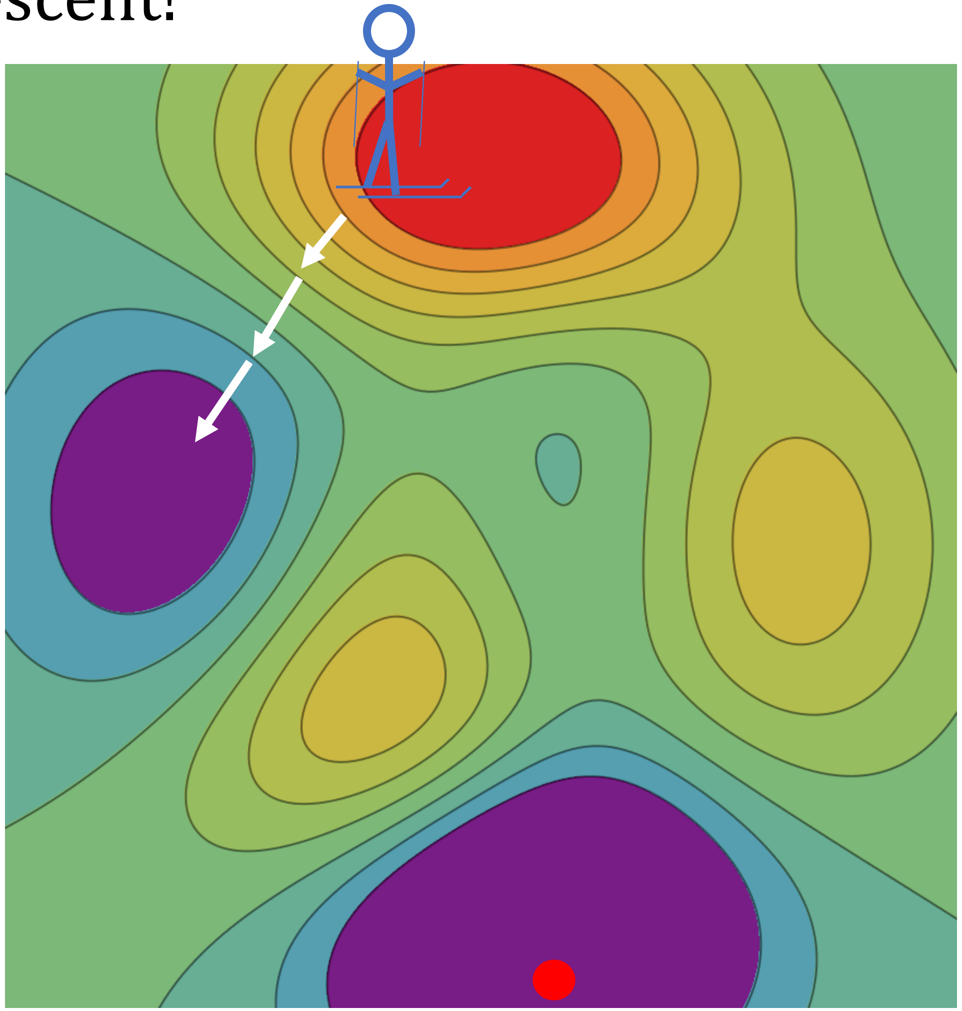

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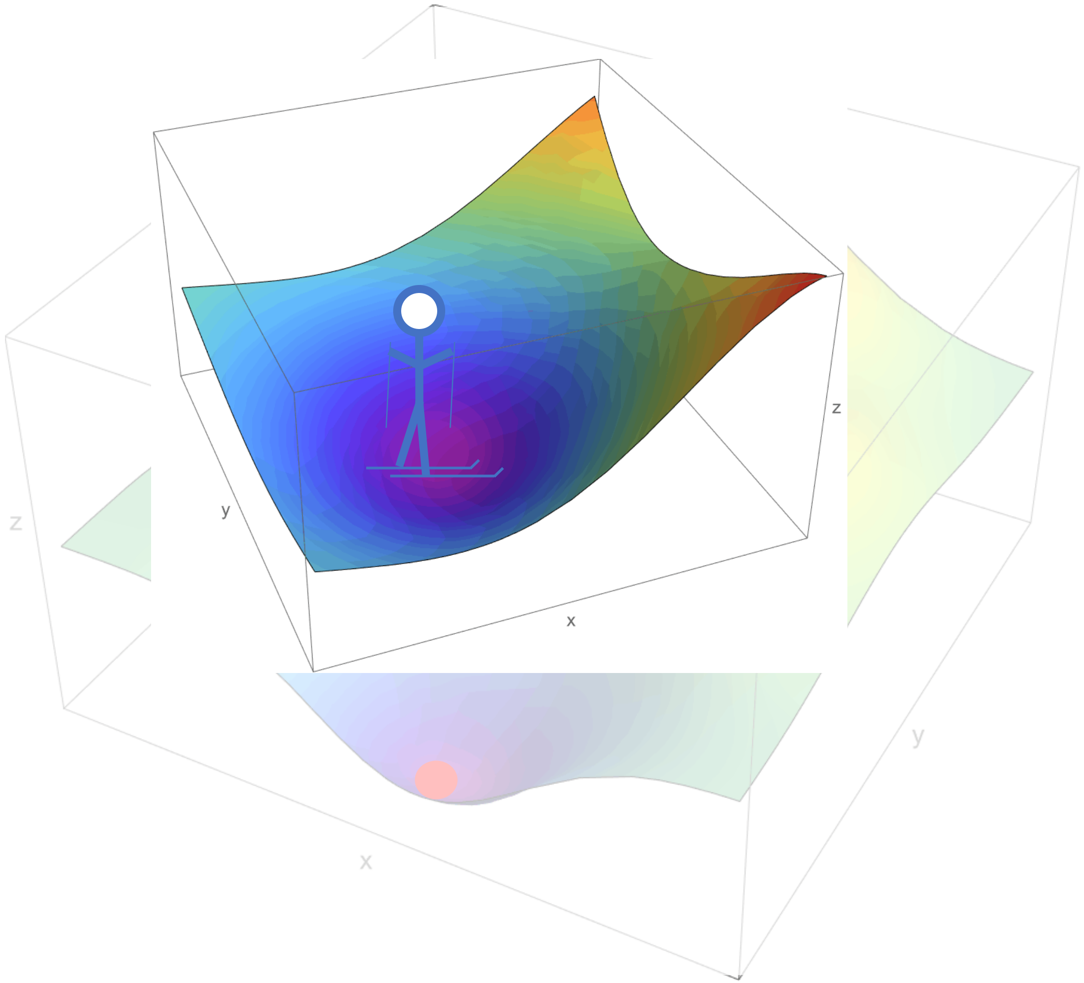

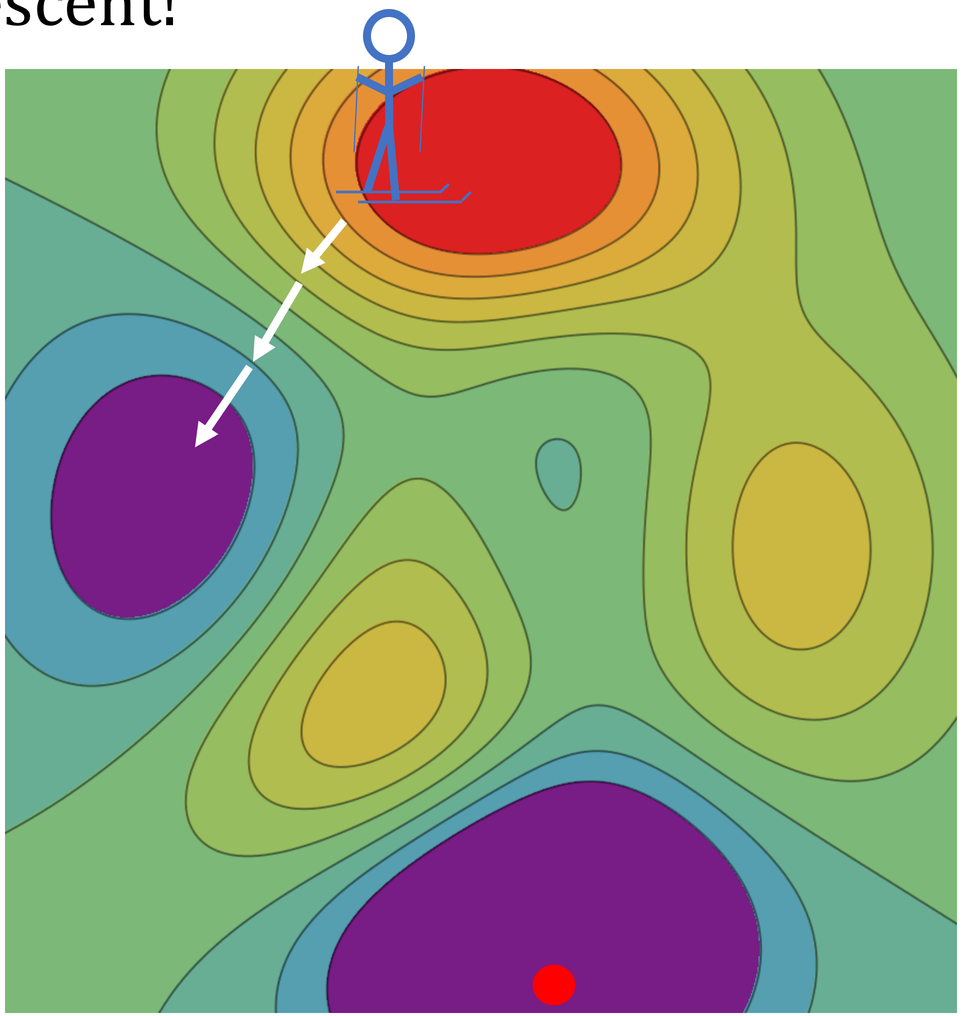

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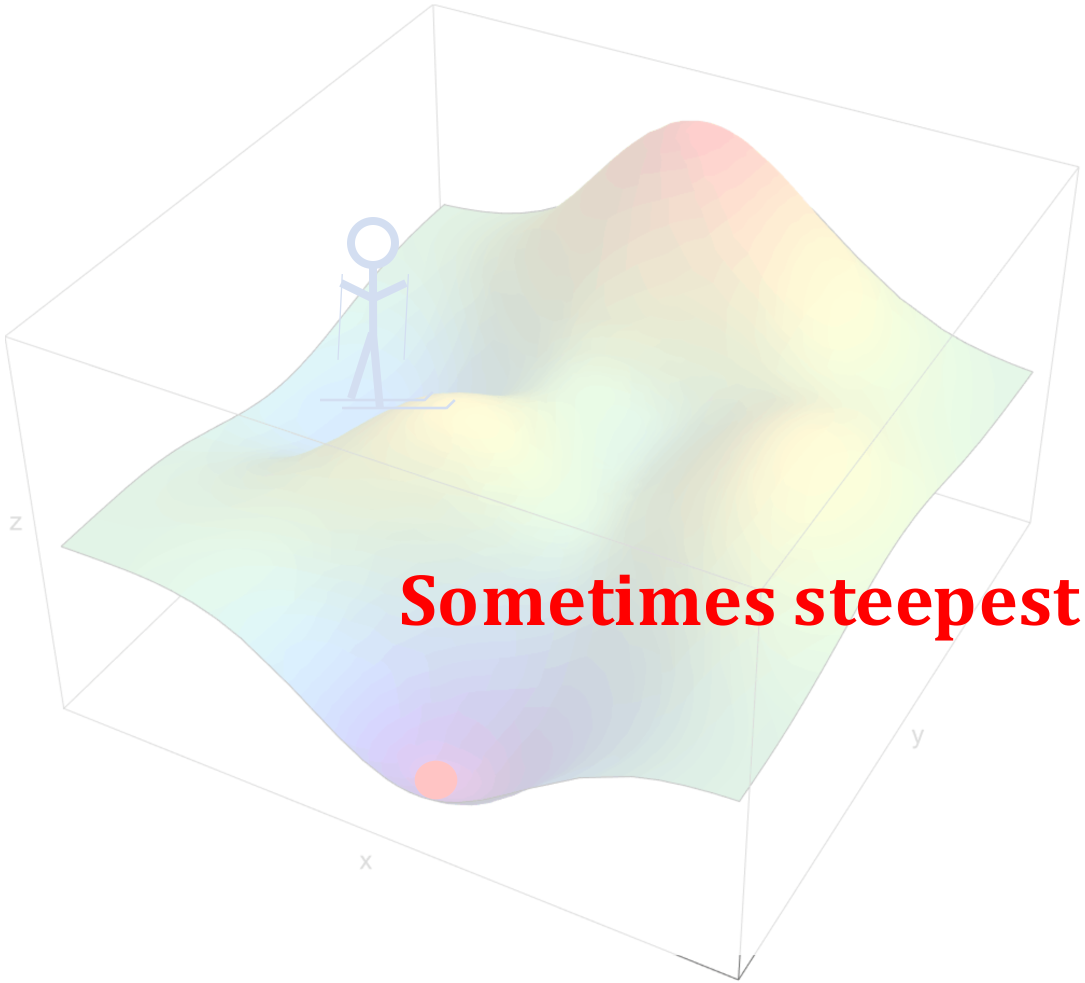

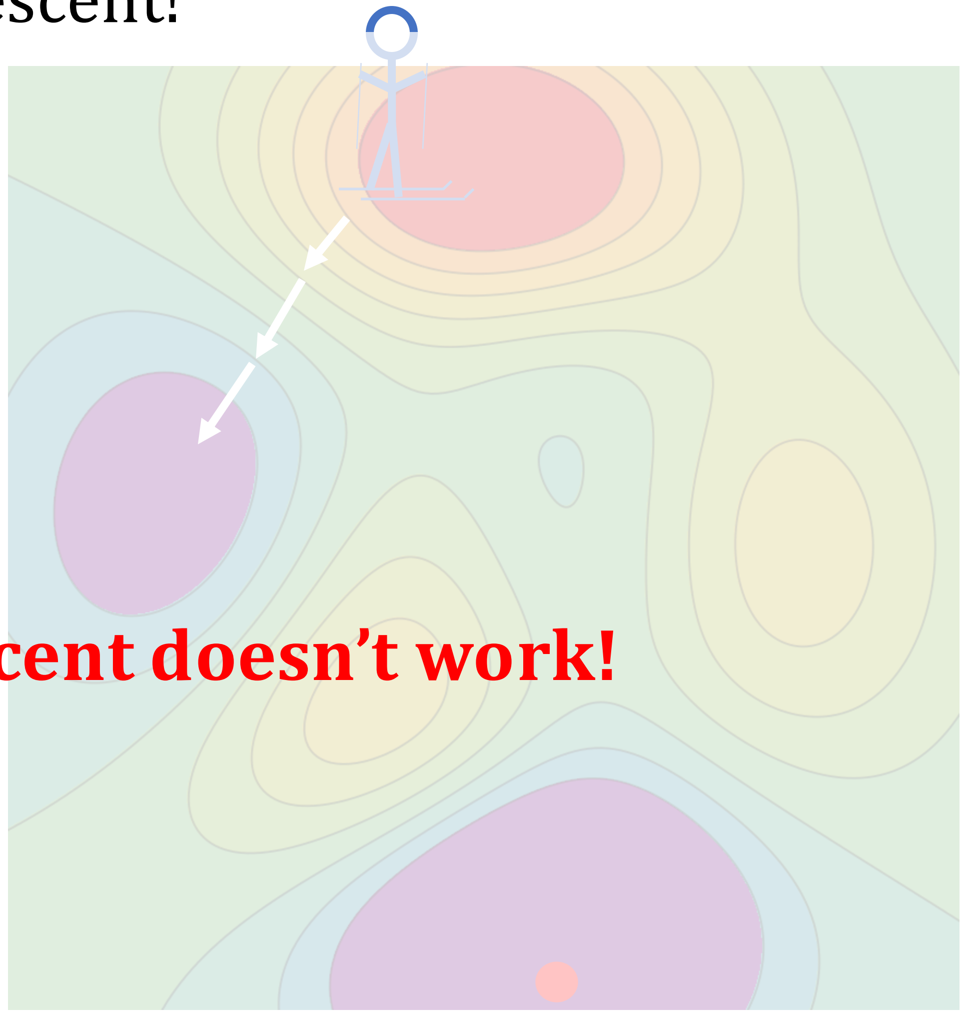

Sometimes steepest descent doesn't work!

Steepest Descent

Q: For which optim problems does steepest descent work?

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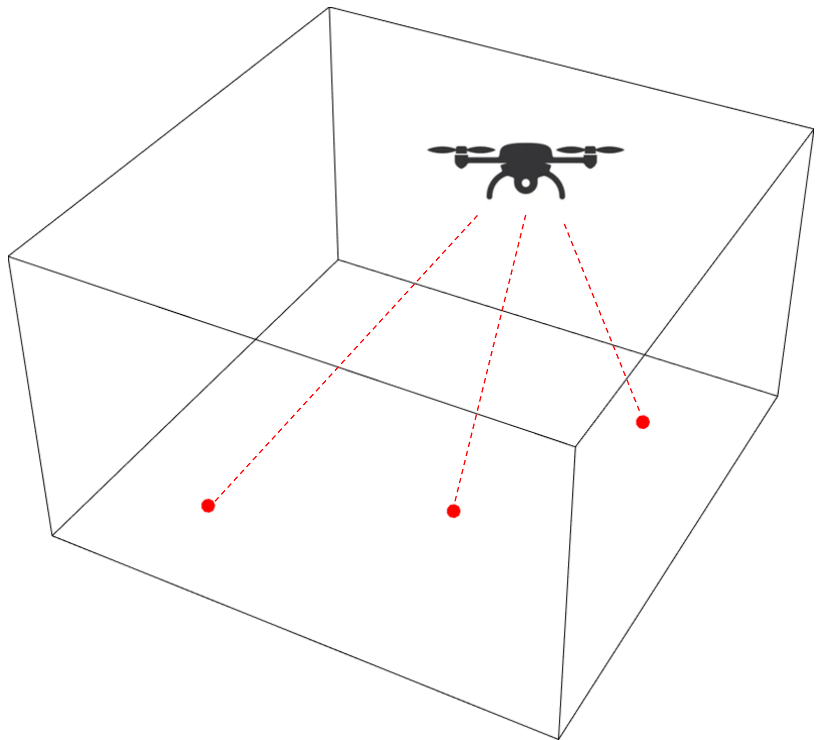

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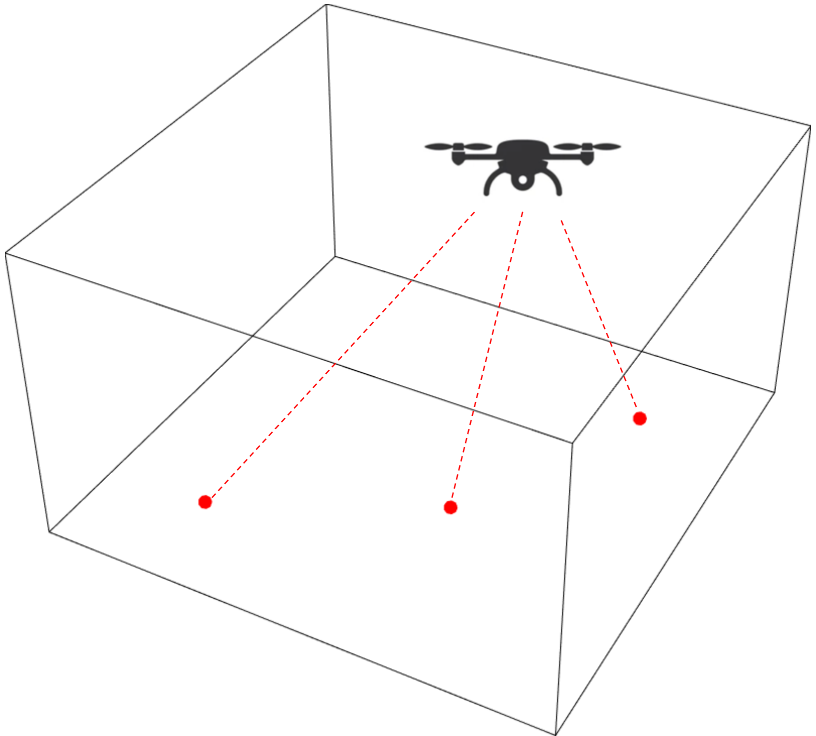
Given: distances between **drone** and **sensors**

Goal: Find location of drone, i.e., x, y, z-coordinates

$$\min \sum_{ij \in E} \left(\|z_i - z_j\|^2 - d_{ij}^2 \right)^2,$$

over $z_1, z_2, \dots, z_n \in \mathbb{R}^k$



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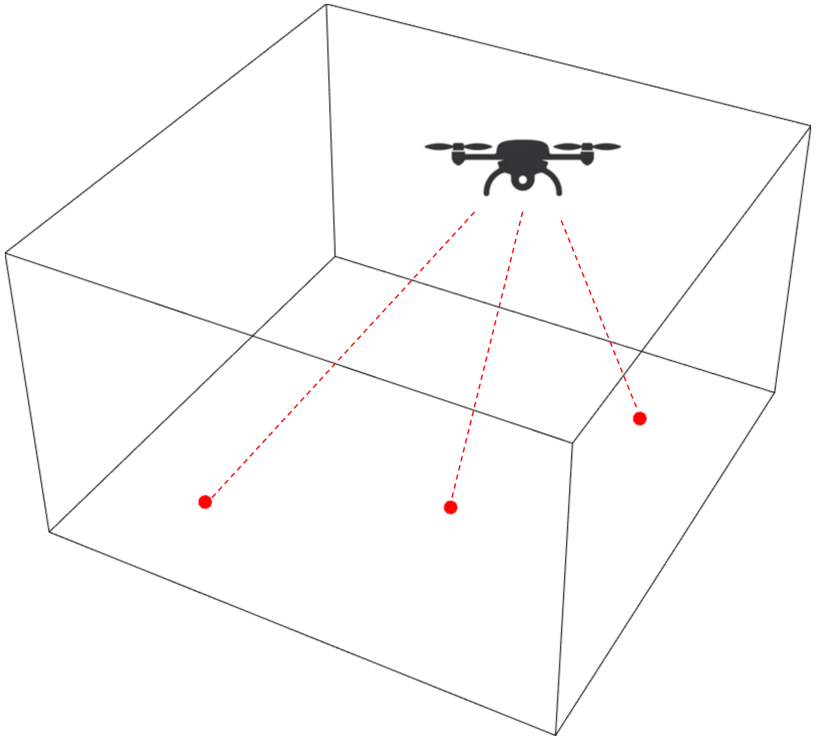

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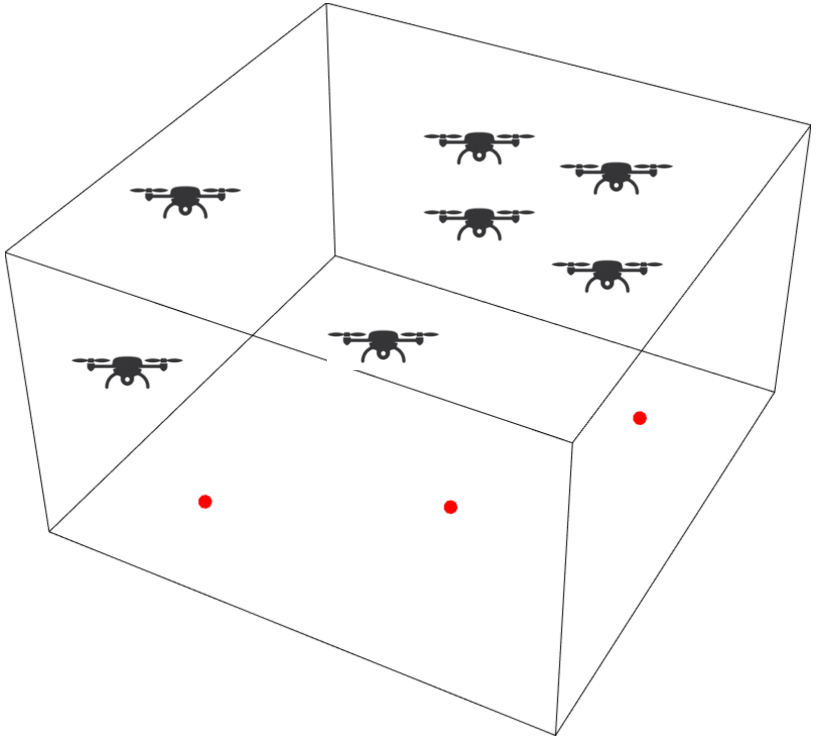
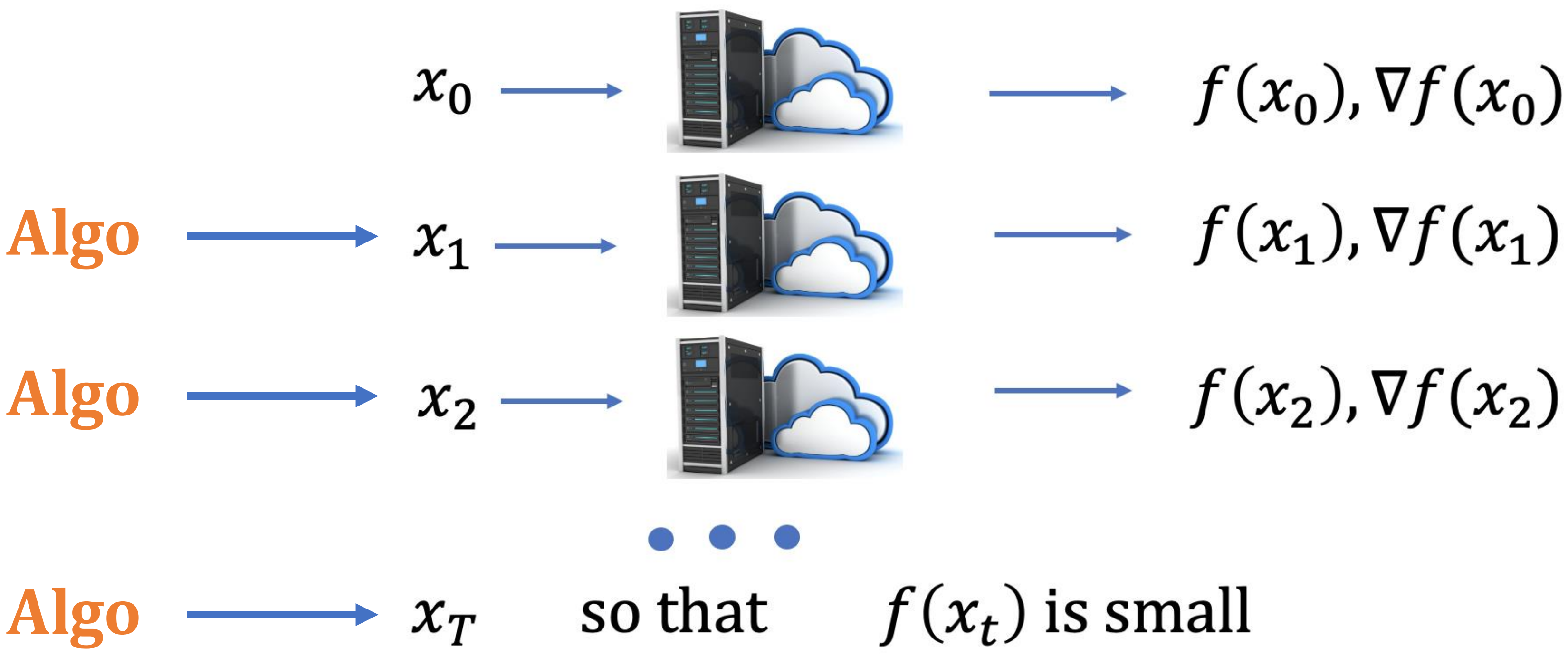


Best algorithm?

Other algorithms than steepest descent?



Best algorithm?

Other algorithms than steepest descent?

Yes! SGD, Nesterov acceleration, Trust regions, …

$x_0 \longrightarrow$  $\longrightarrow f(x_0), \nabla f(x_0)$

Algo $\longrightarrow x_1 \longrightarrow$  $\longrightarrow f(x_1), \nabla f(x_1)$

Algo $\longrightarrow x_2 \longrightarrow$  $\longrightarrow f(x_2), \nabla f(x_2)$

• • •

Algo $\longrightarrow x_T$ so that $f(x_t)$ is small

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Which algo is **best**? [Smallest time T to finish. Eg, assume 1 query per second]

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Which algo is **best**? [Smallest time T to finish. Eg, assume 1 query per second]

- Space of algos is huge (infinite), can't test all of them on your problem!
- That's where math enters – ***prove*** there is no better algorithm

Best algorithm?

Consider a **class** of optimization problems $\{f_1, f_2, \ldots\}$

$$\min_{x \in M} f_i(x)$$

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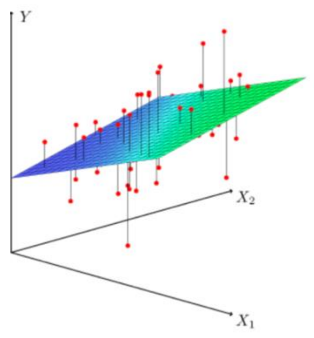

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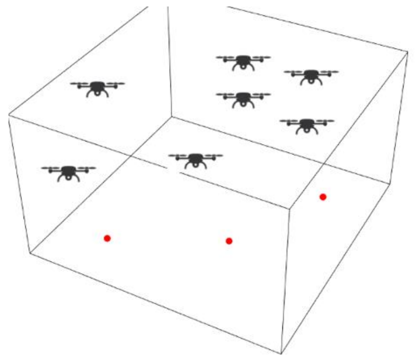
Examples: Linear regression with different sets of data,

Drone localization with different distance measurements



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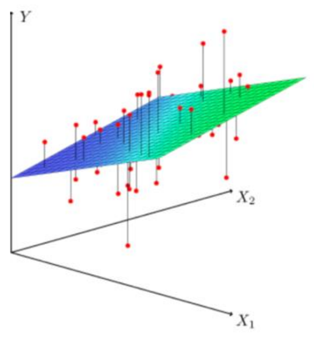

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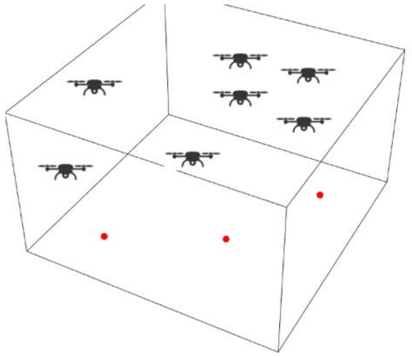

Amanda, Chris, Joe each have an algo

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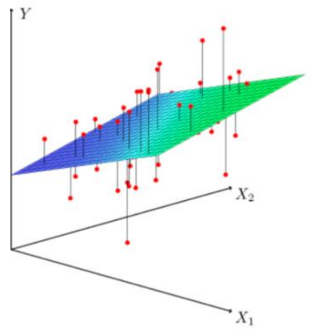

$$\min_{x \in M} f_i(x)$$

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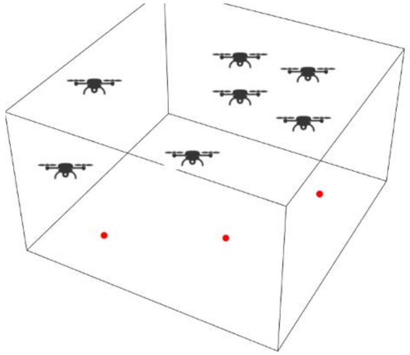

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Amanda: Mine minimizes **every** problem in the **class** in at most 200 seconds.

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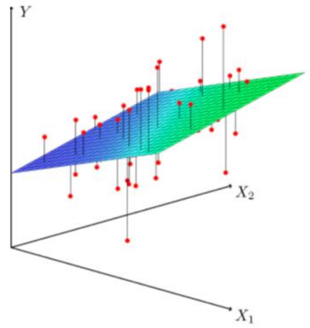

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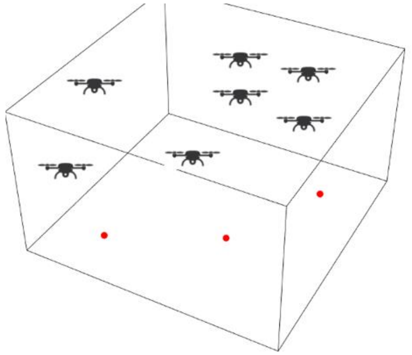

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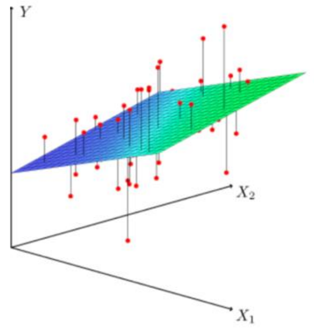

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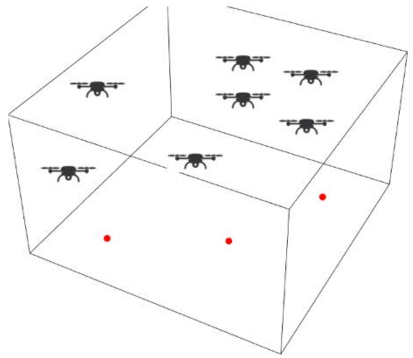

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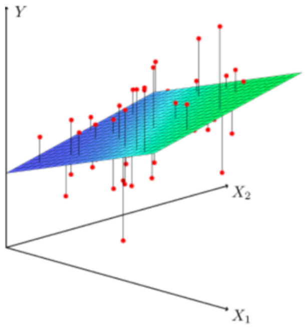
Some consequences:

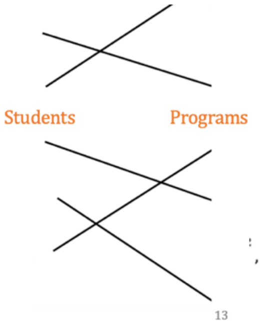
- Steepest descent is optimal for that **class** (under some assumptions)
- Surprising because expectation in community was that steepest descent is not optimal

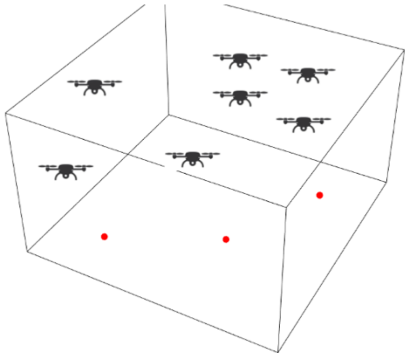
Steepest Descent	Nesterov Accelerated Descent?
Optimal!	**Not Possible!**

Some takeaways

Optimization everywhere







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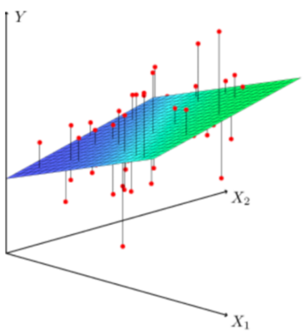
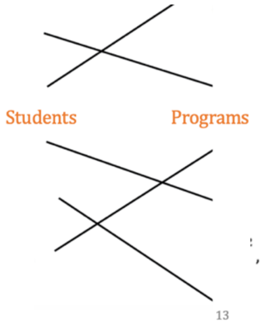
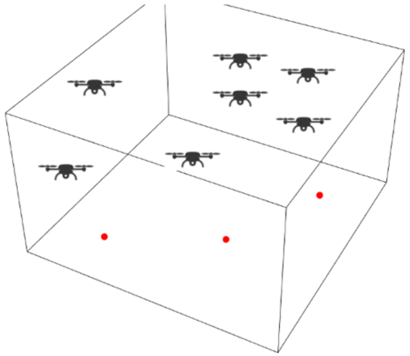
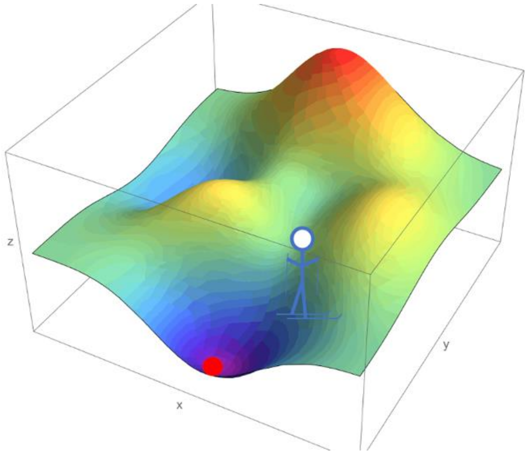
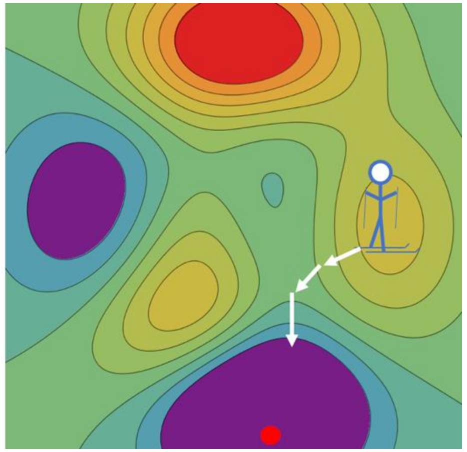






Steepest descent





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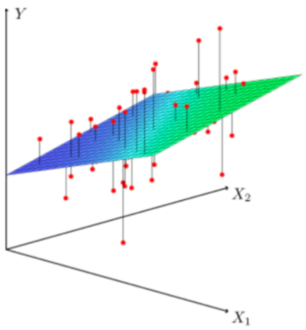

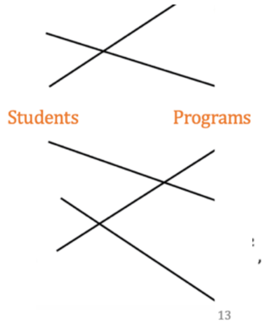

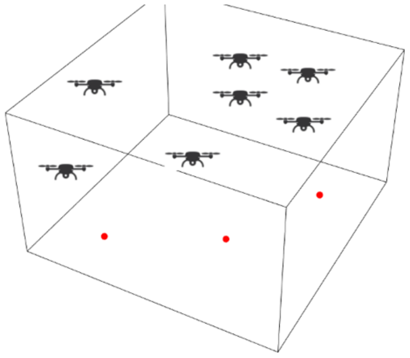

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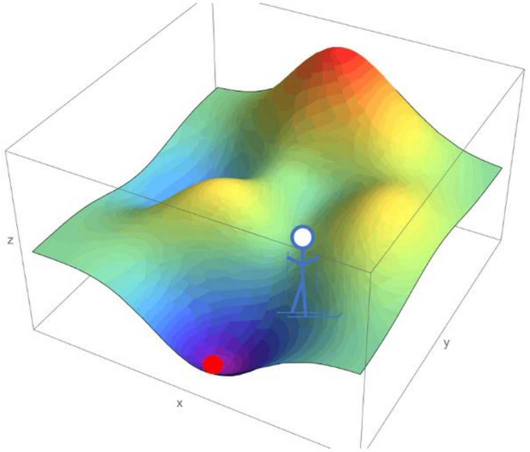

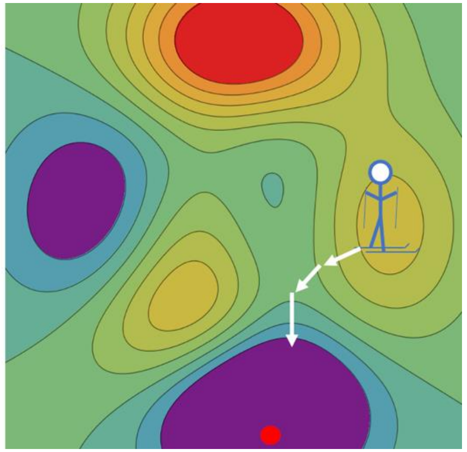

Math in optim: best algorithm

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